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The modified Lobatto–Chebyshev method applied to the numerical evaluation of stress intensity factors in cracks with a corner point

Nikolaos I. Ioakimidis and Pericles S. Theocaris

Department of Theoretical and Applied Mechanics,
The National Technical University of Athens, Greece

Abstract A numerical technique for the evaluation of the stress intensity factors at the tips of a crack with a corner point in plane isotropic elasticity is proposed. This technique consists in using the method of complex Cauchy type singular integral equations and applying the Lobatto–Chebyshev method for their numerical solution. Both cases when the crack arms are of comparable lengths and when the ratio of their lengths is very small or very high can be treated. Some numerical results illustrate the efficiency of the proposed method.

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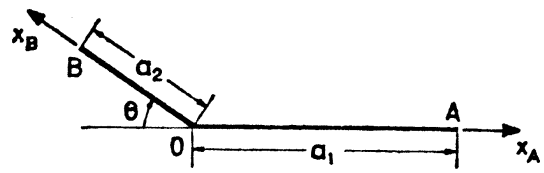

Fig. 1: A crack with a corner point in an infinite medium

1. Introduction

In this technical report, the problem of a crack with a corner point (as in Fig. 1) inside an infinite isotropic elastic medium under conditions of generalized plane stress or plane strain will be considered and special emphasis will be put on the numerical evaluation of the stress intensity factors at its tips. This problem seems to be of some interest since lots of time it is probable that a branching occurs at one of the tips of an initially straight crack. Then there results a crack with two arms as the one shown in Fig. 1. Because of its practical applications, this problem has been already studied by Chatterjee [1] (and by other researchers mentioned in Ref. [1]) as well as by Gupta [2].

The method used by Chatterjee [1] for the solution of the problem under consideration is based on the complex potentials technique of Muskhelishvili [3] and consists in the conformal mapping of the whole crack on the unit circle, the reduction of the problem to a system of integral equations and the numerical solution of this system. This method, to these authors' opinion, seems to be a little complicated and requires a lot of algebraic manipulations mainly due to the conformal mapping of the initial crack on the unit circle. Moreover, the other methods for the solution of the same problem mentioned in Ref. [1] seem to present analogous difficulties.

On the other hand, Gupta [2] reduced the whole problem to a system of four real Cauchy type singular integral equations having formulated this problem in a direct way by distributing edge dislocations along the crack arms in the physical plane. The numerical solution of this system of equations was made by using the Gauss–Jacobi method as this method was applied to the numerical solution of Cauchy type singular integral equations by Erdogan, Gupta and Cook [4].

It seems that the method of Gupta for the solution of the present problem [2] is more practical and clear in its application since it is free from the conformal mapping technique and the corresponding lengthy algebraic manipulations contained in Ref. [1]. Nevertheless, it can also be improved on some points and this will be made in this technical report. These points are

(i) The kernels of the system of the four Cauchy type singular integral equations derived in Ref. [2] are too complicated. One can easily obtain an equivalent system of two complex Cauchy type singular integral equations with very simple complex kernels as will be seen in the next section.

(ii) The application of the Gauss–Jacobi method to the numerical solution of Cauchy type singular integral equations as this method was proposed in Ref. [4] requires the computation of the corresponding abscissae (nodes), weights and collocation points. Such a computation is not very easy. Moreover, the selection of the collocation points proposed in Ref. [4] is not the best one as was shown in Ref. [5] and the convergence of the numerical results is too slow [5]. Additionally, it is required to apply an extrapolation technique for the estimation of the stress intensity factors at the crack tips A and B (Fig. 1), which are, in most cases, the only quantities of interest. Thus, it is preferable to use the Lobatto–Chebyshev method [6] for the numerical solution of the system of Cauchy type singular integral equations and the estimation of the stress intensity factors at the crack tips. The discussion contained in Ref. [7] on the accuracy of the numerical results near the corner point O clearly implies that it is not advantageous at all to apply the Gauss–Jacobi method instead of the Lobatto–Chebyshev method, which assumes a singularity of order $-1/2$, which is smaller than the correct one near the corner point O.

(iii) Finally, the method of Gupta [2] seems ineffective in cases where the ratio of the lengths of the crack arms is too high or too small. In fact, in this reference, numerical results for the stress intensity factors were presented only in cases where this ratio was equal to 1. These results were found to be in agreement with the corresponding results of Ref. [1]. Yet, this is not the case when one of the crack arms is too small compared to the other or inversely. In this case, the numerical results obtained by the method of conformal mapping of Ref. [1] seem not reproducible by the more direct method of Ref. [2]. The method proposed here will be seen to be capable to produce accurate numerical results even in this case. The developments of Ref. [8], where a modification of the Lobatto–Chebyshev method was presented, are also taken here into account.

Furthermore, Wu [9] recently considered elasticity problems of slender Z-cracks incorporating the extreme case where one of the crack arms is much smaller than the other discussed previously. The treatment of the problems considered in Ref. [9] was based on the conformal mapping technique as was also made in Ref. [1]. Nevertheless, to these authors' opinion, the highly complicated formulae used in this reference and the lack of any numerical results make the technique of Ref. [9] quite uncomfortable for practical use.

In the sequel, the method of complex Cauchy type singular integral equations for the formulation and solution of crack problems [10, 11] will be applied to the solution of the problem of a crack with a corner point. This method is simple, versatile and sufficiently accurate to be proved effective in all practical applications. This method is not restricted to the case of straight crack arms (Fig. 1). In fact, the developments of Ioakimidis [11] make its application to the case of curvilinear crack arms a routine procedure.

2. The system of singular integral equations

Consider the crack shown in Fig. 1 consisting of a main straight arm OA of length a_1 and a straight branch OB of length a_2. The angle between the direction of extension of the main arm OA of the crack at the corner point O and the branch OB at this point is denoted by θ. If we consider the common point O of the two arms of the crack as the centre of two coordinate systems Ox_Ay_A and Ox_By_B, the axes Ox_A and Ox_B of which are directed along the arms OA and OB of the crack, respectively, then it is easy to see, if the developments of Panasyuk, Datsyshin and Savruk [10] are taken into account, that the whole problem can be reduced to the following system of two complex Cauchy type singular integral equations:

$$\fint_0^1 \frac{g_i(\tau)}{\tau - t}\,\mathrm{d}\tau + \int_0^1 \Big\{ \big[\overline{S_i(\tau,\lambda_i t)} + e^{2i\theta_i} S_i(\tau,\lambda_i t) \big]\, g_{3-i}(\tau) + \Big[S_i(\tau,\lambda_i t) - e^{2i\theta_i} \frac{S_i^2(\tau,\lambda_i t)}{\overline{S_i(\tau,\lambda_i t)}} \Big] \overline{g_{3-i}(\tau)} \Big\}\,\mathrm{d}\tau = \pi p_i(t), \quad i = 1,2, \quad 0 < t < 1, \tag{1}$$

where

$$\lambda_1 = \frac{a_1}{a_2}, \quad \lambda_2 = \frac{a_2}{a_1}, \quad \theta_1 = \pi - \theta, \quad \theta_2 = \theta - \pi, \quad S_i(\tau,t) = \frac{1}{2(\tau - te^{i\theta_i})}, \quad i = 1,2, \tag{2}$$

and the right-hand side functions $p_i(t)$ $(i = 1,2)$ denote the complex loading of the two arms of the crack consisting of the normal loading $\sigma_{ni}(t)$ and the shear loading $\sigma_{ti}(t)$ on these two arms, that is

$$p_i(t) = \sigma_{ni}(t) + i\sigma_{ti}(t), \quad i = 1,2, \tag{3}$$

which are assumed to be the same on both edges of the crack. We can also remark that the unknown functions $g_{1,2}(t)$ on the arms OA and OB of the crack, respectively (Fig. 1), are proportional to the edge dislocation densities on these arms [10].

Equations (1) should be supplemented by a condition assuring the single-valuedness of displacements around the crack. In the problem under consideration, this condition can be written as

$$\int_0^1 g_1(\tau)\,\mathrm{d}\tau + \lambda_2 e^{i\theta_1} \int_0^1 g_2(\tau)\,\mathrm{d}\tau = 0. \tag{4}$$

Once the system of Cauchy type singular integral equations (1) is numerically solved, obviously with the condition (4) taken into account during its solution, the complex potentials $\Phi(z)$ and $\Psi(z)$ of Muskhelishvili [3] can easily be calculated [10] and, in this way, any quantity of interest, such as the stress and displacement components, can be computed in the whole isotropic elastic plane.

3. Some numerical results

Before we attempt the numerical solution of the system of Cauchy type singular integral equations (1) together with the condition (4), we have also to take into account that the unknown functions $g_{1,2}(t)$ present singularities of order $-1/2$ at the crack tips A and B as is well known in the theory of cracks [11]. At the corner point O these functions also present singularities, as is also well known [12], of order greater than $-1/2$ and smaller than 0. Under these conditions, it seems that the Gauss–Jacobi numerical integration rule could be successfully used for the reduction of Eqs. (1) and (4) to a system of linear algebraic equations in accordance with the developments of Refs. [5, 11]. Nevertheless, if only the stress intensity factors at the crack tips A and B are of interest, then it is possible to use the Lobatto–Chebyshev numerical integration rule for such a reduction assuming, with almost no increase in the final error, that the singularities of the unknown functions $g_{1,2}(t)$ near the corner point O are of order $-1/2$. The Lobatto–Chebyshev numerical integration rule was applied to the numerical solution of Cauchy type singular integral equations for the first time by the present authors [6] and it has the advantage of the direct and very accurate computation of stress intensity factors at crack tips, which are included among the abscissae (nodes) used.

In accordance with these arguments, we can impose the additional condition that one of the functions $f_i(t)$ defined by

$$f_i(t) := \sqrt{t(1-t)}\,g_i(t), \quad i = 1,2, \tag{5}$$

is equal to zero at the corner point O, where $t = 0$. This seems quite reasonable since the order of singularity of the unknown functions $g_{1,2}(t)$ is in reality greater than $-1/2$ at the corner point O. Of course, it is also possible to take into account the theoretical result

$$g_2(t) = -g_1(t)e^{i\varepsilon\lambda\phi} \quad \text{with} \quad \varepsilon := -\operatorname{sign}\theta \tag{6}$$

as $t \to 0$ near the corner point O, where

$$\phi = \pi + |\theta| \tag{7}$$

and λ is the correct value of the order of singularity at the corner point O being the smallest (but greater than -1) root of the transcendental equation [12]

$$\sin[(\lambda+1)\phi] + (\lambda+1)\sin\phi = 0. \tag{8}$$

In reality, since there is no possibility to obtain acceptable numerical results near the point O [7], the use of Eqs. (6)–(8) is not necessary; one can simply use one of Eqs. (5) instead of these equations. In practice, it was seen that the use of one or the other of Eqs. (5) or even Eqs. (6)–(8) had no influence on the obtained numerical results except on those concerning the corner point O itself.

On the basis of all the above developments, the values of the complex stress intensity factors $K_{\mathrm{I}A} + iK_{\mathrm{II}A}$ and $K_{\mathrm{I}B} + iK_{\mathrm{II}B}$ at the crack tips A and B, respectively, were computed for several crack

geometries under a tensile loading $\sigma_\infty = 1$ normal to the main crack OA (Fig. 1) at infinity. In the special case when the arm lengths a_1 and a_2 are equal to 1, $a_1 = a_2 = 1$, the results obtained by the method proposed here are shown in Table 1 for comparison purposes together with the corresponding results given in Ref. [1]. The number n of the abscissae used during numerical integrations was $n = 10$ and $n = 15$ and from Table 1 it can be seen that the results obtained by the present method are of comparable accuracy to those found in Ref. [1] by using the method of conformal mapping.

We can also mention that since the integration interval in Eqs. (1) and (4) is the interval $[0,1]$ instead of the original integration interval $[-1,1]$, here we have to modify the selection of the abscissae (nodes) t_i^*, the weights A_i^* and the collocation points x_k^* of the Lobatto–Chebyshev method [6], which are given by the formulae

$$t_i^* = \cos\frac{(i-1)\pi}{2(n-1)}, \quad i = 1,2,\ldots,2n-1, \tag{9}$$

$$A_1^* = A_{2n-1}^* = \frac{\pi}{4(n-1)}, \quad A_i^* = \frac{\pi}{2(n-1)}, \quad i = 2,3,\ldots,2n-2, \tag{10}$$

$$x_k^* = \cos\frac{(2k-1)\pi}{4(n-1)}, \quad k = 1,2,\ldots,2n-2, \tag{11}$$

by using the simple formulae

$$t_i = 1 - t_i^{*2}, \quad i = 1,2,\ldots,n, \tag{12}$$

$$A_i = 2A_i^*, \quad i = 1,2,\ldots,n-1, \quad A_n = A_n^*, \tag{13}$$

$$x_k = 1 - x_k^{*2}, \quad k = 1,2,\ldots,n-1, \tag{14}$$

which provide the abscissae t_i, the weights A_i and the collocation points x_k actually used in the present numerical integrations.

It was further seen that since in the vicinity of the corner point O the stress field is singular and of a sufficiently complicated nature, it is advisable to increase the density of the abscissae and the collocation points near the corner point O without increasing the number n of the abscissae used. This can easily be achieved by using the results of Ref. [8] concerning such a modification of the Lobatto–Chebyshev method of numerical solution of Cauchy type singular integral equations. Therefore, by using a variable transformation depending on a constant c, e.g. the transformation

$$y = \frac{\tanh cy^*}{\tanh c}, \tag{15}$$

it is possible to select the new abscissae t_i^{**}, the new weights A_i^{**} and the new collocation points x_k^{**} in the present case by using the formulae

$$t_i^{**} = \frac{\tanh ct_i^*}{\tanh c}, \quad i = 1,2,\ldots,2n-1, \tag{16}$$

$$A_i^{**} = \frac{A_i^*}{\sqrt{\tanh c}\,\cosh c}, \quad i = 1, 2n-1, \quad \text{and}$$

$$A_i^{**} = \frac{A_i^*}{\cosh^2 ct_i^*}\sqrt{\frac{1-t_i^{*2}}{\tanh^2 c - \tanh^2 ct_i^*}}, \quad i = 2,3,\ldots,2n-2, \tag{17}$$

$$x_k^{**} = \frac{\tanh cx_k^*}{\tanh c}, \quad i = 1,2,\ldots,2n-2, \tag{18}$$

Table 1

Stress intensity factors at the tips of the crack of Fig. 1 obtained by using the Lobatto–Chebyshev method in its original form with $n = 10$ and $n = 15$ and with $a_1 = a_2 = 1$, $\theta = 15°, 30°, \ldots, 90°$ and $\sigma_\infty = 1$

	$K_{\mathrm{I}A}$	$K_{\mathrm{II}A}$	$K_{\mathrm{I}B}$	$K_{\mathrm{II}B}$
		$\theta = 15°$		
$n = 10$	1.7510	0.0286	1.6618	−0.4777
$n = 15$	1.7511	0.0287	1.6619	−0.4777
Ref. [1]	1.7512	0.0287	1.6619	−0.4777
		$\theta = 30°$		
$n = 10$	1.6926	0.0405	1.3569	−0.8527
$n = 15$	1.6928	0.0406	1.3571	−0.8528
Ref. [1]	1.6930	0.0407	1.3573	−0.8528
		$\theta = 45°$		
$n = 10$	1.6122	0.0255	0.9316	−1.0498
$n = 15$	1.6124	0.0258	0.9319	−1.0498
Ref. [1]	1.6127	0.0261	0.9322	−1.0499
		$\theta = 60°$		
$n = 10$	1.5284	−0.0155	0.4864	−1.0402
$n = 15$	1.5282	−0.0149	0.4866	−1.0396
Ref. [1]	1.5283	−0.0145	0.4865	−1.0392
		$\theta = 75°$		
$n = 10$	1.4567	−0.0718	0.1209	−0.8481
$n = 15$	1.4556	−0.0706	0.1210	−0.8455
Ref. [1]	1.4547	−0.0701	0.1203	−0.8429
		$\theta = 90°$		
$n = 10$	1.4048	−0.1284	−0.0922	−0.5437
$n = 15$	1.4013	−0.1250	−0.0911	−0.5360
Ref. [1]	1.3984	−0.1230	−0.0921	−0.5269

Table 2

Stress intensity factors at the tips of the crack of Fig. 1 obtained by using the Lobatto–Chebyshev method in its modified form with $c_1 = c_2 = 0, 1, \ldots, 4$, with $n = 10$ and $n = 15$ and with $a_1 = a_2 = 1$, $\theta = 45°$ and $\theta = 90°$ and $\sigma_\infty = 1$

$c_{1,2}$	n	K_{IA}	K_{IIA}	K_{IB}	K_{IIB}
			$\theta = 45°$		
0	10	1.6122	0.0255	0.9316	−1.0498
0	15	1.6124	0.0258	0.9319	−1.0498
1	10	1.6123	0.0258	0.9318	−1.0498
1	15	1.6125	0.0259	0.9320	−1.0498
2	10	1.6125	0.0260	0.9321	−1.0498
2	15	1.6126	0.0260	0.9321	−1.0499
3	10	1.6126	0.0261	0.9322	−1.0499
3	15	1.6126	0.0261	0.9322	−1.0499
4	10	1.6125	0.0261	0.9322	−1.0499
4	15	1.6126	0.0261	0.9322	−1.0499
Ref. [1]		1.6127	0.0261	0.9322	−1.0499
			$\theta = 90°$		
0	10	1.4048	−0.1284	−0.0922	−0.5437
0	15	1.4013	−0.1250	−0.0911	−0.5360
1	10	1.4023	−0.1258	−0.0913	−0.5381
1	15	1.4001	−0.1241	−0.0909	−0.5332
2	10	1.3995	−0.1239	−0.0909	−0.5320
2	15	1.3987	−0.1238	−0.0910	−0.5302
3	10	1.3985	−0.1239	−0.0911	−0.5297
3	15	1.3981	−0.1240	−0.0912	−0.5290
4	10	1.3985	−0.1254	−0.0915	−0.5297
4	15	1.3979	−0.1242	−0.0913	−0.5285
Ref. [1]		1.3984	−0.1230	−0.0921	−0.5269

instead of Eqs. (9)–(11) before the use of Eqs. (12)–(14) (together with Eqs. (15)–(18)) in order to find the values of the abscissae t_i, the weights A_i and the collocation points x_k which will be really used during the numerical solution of the system of Eqs. (1) and (4). Of course, it is possible that the constant c in Eqs. (15)–(18) takes two different values, c_1 and c_2, for the two crack arms OA and OB (Fig. 1), respectively.

In Table 2, the numerical results for the stress intensity factors at the crack tips obtained by applying this technique to the crack of Fig. 1 with $a_1 = a_2 = 1$, $\sigma_\infty = 1$ (and of direction normal to the branch OA) and $\theta = 45°$ and $90°$ are presented. The two constants c_1 and c_2 were assumed to

Table 3

Stress intensity factors at the tips of the crack of Fig. 1 obtained by using the Lobatto–Chebyshev method in its original form for the arm OB and in its modified form with $c_1 = 0, 1, \ldots, 4$ for the arm OA, with $n = 10$ and $n = 15$ and with $a_1 = 1$, $a_2 = 0.005$, $\theta = 45°$ and $\theta = 90°$ and $\sigma_\infty = 1$

c_1	n	K_{IA}	K_{IIA}	K_{IB}	K_{IIB}
			$\theta = 45°$		
0	10	1.2555	0.0020	1.1067	−0.7540
0	15	1.2555	0.0019	0.9947	−0.5462
1	10	1.2555	0.0019	1.0209	−0.5970
1	15	1.2555	0.0018	0.9592	−0.4861
2	10	1.2556	0.0018	0.9451	−0.4705
2	15	1.2556	0.0017	0.9380	−0.4675
3	10	1.2555	0.0017	0.9380	−0.4664
3	15	1.2556	0.0017	0.9379	−0.4652
4	10	1.2552	0.0017	0.9393	−0.4643
4	15	1.2556	0.0017	0.9386	−0.4661
Ref. [1]				0.939	−0.465
			$\theta = 90°$		
0	10	1.2533	0.0026	0.3808	−0.9369
0	15	1.2536	0.0019	0.4531	−0.5657
1	10	1.2535	0.0021	0.4429	−0.6647
1	15	1.2539	0.0015	0.4245	−0.4389
2	10	1.2540	0.0015	0.3782	−0.4182
2	15	1.2541	0.0014	0.3536	−0.4505
3	10	1.2540	0.0014	0.3514	−0.4433
3	15	1.2541	0.0014	0.3482	−0.4472
4	10	1.2537	0.0014	0.3528	−0.4662
4	15	1.2541	0.0014	0.3504	−0.4600
Ref. [1]				0.346	−0.466

be equal: $c = c_1 = c_2 = 0, 1, \ldots, 4$ with the value $c = c_1 = c_2 = 0$ corresponding to the original form of the Lobatto–Chebyshev method. As regards the number of abscissae n, it took the values $n = 10$ and $n = 15$ for both crack arms OA and OB. From the numerical results of Table 2 it can be concluded that the best values for the stress intensity factors at the crack tips A and B are obtained by using the modified form of the Lobatto–Chebyshev method, Eqs. (16)–(18), instead of its original form, Eqs. (9)–(11). Moreover, the values $c = 2$, $c = 3$ and $c = 4$ of the constant c seem to be sufficiently appropriate although the value $c = 4$ is not recommended for use since the densities of the abscissae and the collocation points increase too much near the corner point O and become too small near the crack tips A and B (Fig. 1). Of course, a satisfactory value of the constant $c = c_1 = c_2$

can be found by observing the convergence of the obtained numerical results as n increases. Furthermore, it is quite possible to replace Eq. (15) by some other variable transformation function [8] although this is not believed to lead to a dramatic increase in the accuracy of the numerical results.

Finally, in Table 3, analogous results were derived for the case where $a_1 = 1$ and $a_2 = 0.005$ with $c_1 = 0, 1, \ldots, 4$ and $c_2 = 0$. This case was also considered in Ref. [1] and its practical interest is greater than that of the case considered in Table 1 and Table 2 since it corresponds to the initial stages of crack branching before the growth of the branch, which is sufficiently important in fracture mechanics. In this case, we observe that the use of the Lobatto–Chebyshev method in its modified form for the crack branch OA with $c_1 = 2, 3, 4$ is necessary in order to obtain sufficiently accurate numerical results contrary to what happened in Table 2. Moreover, there is no need to use the modified form of the Lobatto–Chebyshev method on the crack arm OB or to use different values for the number n of abscissae on the crack arms OA and OB. The improvement of the numerical results achieved by such modifications seems to be negligible.

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