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A curvilinear crack along the interface of two plane isotropic elastic media[2]

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Abstract The plane elasticity problem of a curvilinear crack along the interface of two plane isotropic elastic media is treated by using the method of complex potentials of Muskhelishvili. The problem is reduced to a complex Cauchy type singular integral equation along the whole interface of the two media, the crack included. This equation is applied to the simple problem of a straight crack along a straight interface and the obtained results are seen to be in agreement with the already available solutions of this special problem. Finally, a complete description of the method of numerical solution of the complex Cauchy type singular integral equation in its general form is made.

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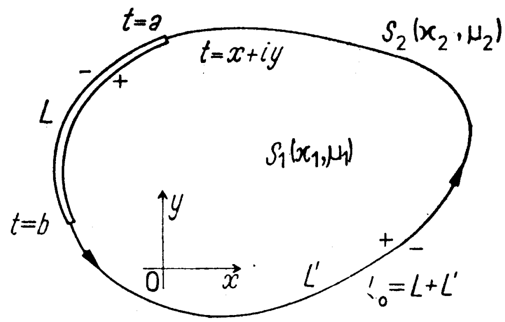

Fig. 1: Geometry of a curvilinear crack along the interface of two plane isotropic elastic media

1. Introduction

The problem of a crack along the interface of two plane isotropic elastic media under plane strain or generalized plane stress conditions has drawn long ago the attention of researchers in plane elasticity because of its practical importance in engineering problems. The first case of this problem which was considered was that of two bonded elastic half-planes (of different materials) with a crack or a system of cracks along their interface [2–10]. Moreover, the problem of a circular (or elliptical) arc crack or a system of such cracks along the interface of a circular (or elliptical) inclusion inside an infinite matrix was solved [11–15]. Because of the simple shape of the crack as well as of the whole interface, these problems possess closed-form solutions obtained by using the complex potential approach of Muskhelishvili [16] and, in general, by reducing the whole problem to a system of Riemann–Hilbert boundary value problems (or problems of linear relationship in the terminology of Ref. [16]), the closed-form solution of which can easily be derived.

Here we will consider the general case of a curvilinear crack along the interface (of arbitrary shape) of two plane isotropic elastic media under quite general loading conditions. This problem will be reduced to a complex Cauchy type singular integral equation along the whole interface with the crack included. The method of treatment of the problem will be the method of complex potentials of Muskhelishvili [16] but contrary to the cases of straight or circular-arc-shaped interface cracks, no possibility of a final reduction of the problem to a system of Riemann–Hilbert boundary value problems exists. The present developments constitute an extension of the previous results of the authors, who solved, by reduction to complex singular integral equations, the problems of a curvilinear crack or a system of such cracks inside a homogeneous infinite isotropic [1, 17–19] or anisotropic [20] medium as well as the problem of an isotropic elastic inclusion of arbitrary shape inside an infinite or finite isotropic elastic matrix [21].

As regards the numerical solution of the complex Cauchy type singular integral equation to which the problem is reduced, the Lobatto–Jacobi method or some other method associated with complex singularities at the end-points of the integration interval can directly be used in accordance with the developments of Refs. [22–24]. Finally, it will be shown that in the case of an interface crack along the straight boundary of two plane isotropic elastic half-planes, the complex Cauchy type singular integral equation is equivalent to the solutions of this special problem already available by reduction of the problem to a system of Riemann–Hilbert boundary value problems.

2. Statement of the problem

We consider the problem of an infinite isotropic elastic medium (matrix) containing an inclusion made of a different isotropic elastic material under plane strain or generalized plane stress conditions. The shape of the common boundary between the matrix and the inclusion (interface) is arbitrary. It is further assumed that the bonding between the matrix and the inclusion along the interface is not perfect and, thus, a curvilinear crack (or a system of such cracks) is formed along the interface. We denote by L_0 the whole boundary between the matrix and the inclusion, by L its part constituting the crack and by L' the remaining part of the boundary between the matrix and the inclusion where a perfect bonding is assumed to exist. The whole geometry of the inhomogeneous medium under consideration is shown in Fig. 1. The materials of the inclusion S_1 and the matrix S_2 are characterized by their shear moduli μ_1 and μ_2 and their Poisson ratios $\nu_1$and ν_2, respectively, or, better, by the constants κ_1 and κ_2, respectively, defined by $\kappa_{1,2} = 3-4\nu_{1,2}$ for plane strain conditions and by $\kappa_{1,2} = (3-\nu_{1,2})/(1+\nu_{1,2})$ for generalized plane stress conditions [16]. Furthermore, the side of the interface towards the inclusion S_1 is denoted by the symbol $+$ while the side of the interface towards the matrix S_2 is denoted by the symbol $-$. Then the direction along the interface L_0 will be anticlockwise as usual. Finally, the loading of the whole medium is assumed to be arbitrary consisting of the loading at infinity, characterized by the constants Γ and Γ' of Muskhelishvili [16], as well as of the loading along the two faces of the crack L described by its normal and shear components $\sigma_n^{\pm}(t) + i\sigma_t^{\pm}(t)$ along the two faces $+$ and $-$ of the crack L. The symbol t as well as the symbol τ denote both the points of the interface L_0 (the crack L included) as well as the complex abscissas of these points t or $\tau = x+iy$, where x and y are the Cartesian coordinates of the points t or τ of the interface L_0 (Fig. 1).

In fact, the loading conditions on the crack L are those of the first boundary value problem in accordance with the terminology used in Ref. [16]. In most practical problems, these are the really existing conditions along the crack L. In any case, if the displacement components are given on both crack faces (second fundamental problem) or a part of the crack faces (mixed fundamental problem), then the present results should be slightly modified as was already made in Ref. [1] for the problem of a simple smooth curvilinear crack inside an infinite isotropic elastic medium. This modification is of a trivial character and it will not concern us here.

We will try to determine the complex potentials $\Phi(z)$ and $\Psi(z)$ of Muskhelishvili, which are the derivatives of the complex potentials $\phi(z)$ and $\psi(z)$, respectively [16]. The advantages of making use of $\Phi(z)$ and $\Psi(z)$ instead of $\phi(z)$ and $\psi(z)$ are explained in Refs. [1, 21]. The boundary conditions that should be satisfied are the following conditions:

(i) The complex potentials $\Phi(z)$ and $\Psi(z)$ should tend to the values Γ and Γ', respectively, as z tends to infinity, that is [16]

$$\Phi(z) = \Gamma + O\left(\frac{1}{z}\right), \quad \Psi(z) = \Gamma' + O\left(\frac{1}{z}\right) \quad \text{as} \quad z \to \infty. \tag{1}$$

(ii) The stress components on the crack L should be those assumed to exist, i.e. the stress components $\sigma_n^{\pm}(t)$ and $\sigma_t^{\pm}(t)$. It is easily seen that this condition can be written as [1]

$$\Phi^{\pm}(t) + \overline{\Phi^{\pm}(t)} + \frac{\mathrm{d}t}{\overline{\mathrm{d}t}}[\bar{t}\Phi'^{\pm}(t) + \Psi^{\pm}(t)] = \sigma_n^{\pm}(t) - i\sigma_t^{\pm}(t), \quad t \in L. \tag{2}$$

Here the symbol $\mathrm{d}t/\overline{\mathrm{d}t}$ is defined as the ratio $(\mathrm{d}t/\mathrm{d}s)/(\overline{\mathrm{d}t}/\mathrm{d}s)$, where s is a real variable varying along the crack L, e.g. the arc length.

(iii) The stress components should be continuous across the part L' of the interface L_0 where the bonding is perfect. This condition, because of Eqs. (2), can be written as [21]

$$\Phi^{+}(t) + \overline{\Phi^{+}(t)} + \frac{\mathrm{d}t}{\overline{\mathrm{d}t}}[\bar{t}\Phi'^{+}(t) + \Psi^{+}(t)] = \Phi^{-}(t) + \overline{\Phi^{-}(t)} + \frac{\mathrm{d}t}{\overline{\mathrm{d}t}}[\bar{t}\Phi'^{-}(t) + \Psi^{-}(t)], \quad t \in L'. \tag{3}$$

(iv) The displacement components u and v should differ on L' by a known quantity described by a function $g(t)$, that is

$$[u^{+}(t) + iv^{+}(t)] - [u^{-}(t) + iv^{-}(t)] = g(t). \tag{4}$$

In terms of the complex potentials $\Phi(z)$ and $\Psi(z)$, Eq. (4) can be written as [21]

$$\Phi^{+}(t)-\kappa_1\overline{\Phi^{+}(t)}+\frac{\mathrm{d}t}{\overline{\mathrm{d}t}}[\bar{t}\Phi'^{+}(t)+\Psi^{+}(t)]$$
$$=\Gamma_0\left\{\Phi^{-}(t)-\kappa_2\overline{\Phi^{-}(t)}+\frac{\mathrm{d}t}{\overline{\mathrm{d}t}}[\bar{t}\Phi'^{-}(t)+\Psi^{-}(t)]\right\}+G(t), \tag{5}$$

where

$$\Gamma_0=\frac{\mu_1}{\mu_2}, \tag{6}$$

$$G(t)=-2\mu_1\overline{\left(\frac{\mathrm{d}g(t)}{\mathrm{d}t}\right)}. \tag{7}$$

(v) Finally, we should pay attention to the condition of single-valuedness of displacements. It can easily be seen [21] that the displacements will be single-valued in the whole plane if the following conditions hold:

$$\oint_{L_0}\mathrm{d}[u^{\pm}(t)+iv^{\pm}(t)]=0. \tag{8}$$

Because of the fact that the complex potentials $\Phi(z)$ and $\Psi(z)$ must be holomorphic functions inside the inclusion S_1 and this will be taken into account in advance in the developments of the next section, the first of the conditions (8) corresponding to the sign $+$ (inclusion S_1) is identically satisfied. Then the conditions (8) can be expressed as

$$\oint_{L_0}\mathrm{d}\{[u^{+}(t)-u^{-}(t)]+i[v^{+}(t)-v^{-}(t)]\}\,\mathrm{d}t=0. \tag{9}$$

By taking into account the formula relating the displacements $u(z)$ and $v(z)$ to the complex potentials $\Phi(z)$ and $\Psi(z)$ [16], we obtain

$$-2\mu_{1,2}\left[\frac{\mathrm{d}u^{\pm}(t)}{\overline{\mathrm{d}t}}-i\frac{\mathrm{d}v^{\pm}(t)}{\overline{\mathrm{d}t}}\right]=\Phi^{\pm}(t)-\kappa_{1,2}\overline{\Phi^{\pm}(t)}+\frac{\mathrm{d}t}{\overline{\mathrm{d}t}}\{\bar{t}\Phi'^{\pm}(t)+\Psi^{\pm}(t)\}. \tag{10}$$

Then, Eq. (9) takes the form

$$\int_{L}\{\{[\Phi^{+}(t)-\Gamma_0\Phi^{-}(t)]-[\kappa_1\overline{\Phi^{+}(t)}-\Gamma_0\kappa_2\overline{\Phi^{-}(t)}]\}\overline{\mathrm{d}t}+\{\bar{t}[\Phi'^{+}(t)-\Gamma_0\Phi'^{-}(t)]$$
$$+[\Psi^{+}(t)-\Gamma_0\Psi^{-}(t)]\}\,\mathrm{d}t\}=-\int_{L'}G(t)\,\overline{\mathrm{d}t}=-2\mu_1[g(b)-g(a)], \tag{11}$$

where $t=a$ and $t=b$ are the end-points of the crack L (Fig. 1).

3. The singular integral equations

Now we will try to reduce our problem, expressed by the boundary conditions (i) to (v) of the previous section, to a complex Cauchy type singular integral equation along the interface L_0. The whole procedure will be analogous to the one used in Refs. [17–21] and it is based on the fact that the complex potentials $\Phi(z)$ and $\Psi(z)$ are sectionally holomorphic functions in the whole complex plane except the interface L_0.

At first, by adding and subtracting Eqs. (2), we obtain

$$[\Phi^{+}(t)+\Phi^{-}(t)]+[\overline{\Phi^{+}(t)}+\overline{\Phi^{-}(t)}]$$
$$+\frac{\mathrm{d}t}{\overline{\mathrm{d}t}}\{\bar{t}[\Phi'^{+}(t)+\Phi'^{-}(t)]+[\Psi^{+}(t)+\Psi^{-}(t)]\}=2\overline{p(t)},\quad t\in L, \tag{12}$$

$$[\Phi^{+}(t)-\Phi^{-}(t)]+[\overline{\Phi^{+}(t)}-\overline{\Phi^{-}(t)}]$$
$$+\frac{\mathrm{d}t}{\overline{\mathrm{d}t}}\{\bar{t}[\Phi'^{+}(t)-\Phi'^{-}(t)]+[\Psi^{+}(t)-\Psi^{-}(t)]\}=2\overline{q(t)},\quad t\in L, \tag{13}$$

where

$$2p(t) = [\sigma_n^+(t) + \sigma_n^-(t)] + i[\sigma_t^+(t) + \sigma_t^-(t)], \tag{14}$$

$$2q(t) = [\sigma_n^+(t) - \sigma_n^-(t)] + i[\sigma_t^+(t) - \sigma_t^-(t)]. \tag{15}$$

Equation (3) can also be written as

$$[\Phi^+(t) - \Phi^-(t)] + [\overline{\Phi^+(t)} - \overline{\Phi^-(t)}] + \frac{\overline{dt}}{dt}\{\bar{t}[\Phi'^+(t) - \Phi'^-(t)] + [\Psi^+(t) - \Psi^-(t)]\} = 0, \quad t \in L'. \tag{16}$$

Since the complex potential $\Phi(z)$ is holomorphic in the whole complex plane except the interface L_0 and it should satisfy the first of Eqs. (1) at infinity, it can be expressed in the form of a Cauchy type integral with an unknown density function $\phi(t)$ on L_0, that is

$$\Phi(z) = \frac{1}{2\pi i}\int_{L_0} \frac{\phi(\tau)}{\tau - z}\,d\tau + \Gamma. \tag{17}$$

Because of the first Plemelj formula [16] and Eqs. (13) and (16), it follows that

$$\Psi^+(t) - \Psi^-(t) = -2\overline{q(t)} - \frac{\overline{dt}}{dt}\,\overline{\phi(t)} - \frac{d}{dt}\{\bar{t}\phi(t)\}, \quad t \in L, \tag{18}$$

$$\Psi^+(t) - \Psi^-(t) = -\frac{\overline{dt}}{dt}\,\overline{\phi(t)} - \frac{d}{dt}\{\bar{t}\phi(t)\}, \quad t \in L'. \tag{19}$$

Then, in accordance with the developments of Ref. [16], it is evident that the appropriate expression of the complex potential $\Psi(z)$ is

$$\Psi(z) = -\frac{1}{\pi i}\int_L \frac{\overline{q(\tau)}}{\tau - z}\,\overline{d\tau} - \frac{1}{2\pi i}\int_{L_0} \frac{\overline{\phi(\tau)}}{\tau - z}\,\overline{d\tau} - \frac{1}{2\pi i}\int_{L_0} \frac{\bar{\tau}\phi(\tau)}{(\tau - z)^2}\,d\tau + \Gamma', \tag{20}$$

where the second of Eqs. (1) was also taken into consideration. Equations (17) and (20) make clear that the determination of the unknown density function $\phi(t)$ in the Cauchy type integrals permits the evaluation of the complex potentials $\Phi(z)$ and $\Psi(z)$ and, furthermore, of the stress and displacement fields inside the whole complex plane, that is both in the inclusion S_1 and in the matrix S_2.

Now, from Eq. (12) and because of Eqs. (17) and (20) and the second Plemelj formula [16], it results in

$$\operatorname{Re}\left\{\frac{1}{\pi i}\int_{L_0} \frac{\phi(\tau)}{\tau - t}\,d\tau\right\} - \frac{dt}{\overline{dt}}\frac{1}{\pi i}\int_{L_0} \frac{\operatorname{Re}[(\bar{\tau} - \bar{t})\phi(\tau)\,d\tau]}{(\tau - t)^2} = \overline{p(t)} - \frac{dt}{\overline{dt}}\frac{1}{\pi i}\int_L \frac{\overline{q(\tau)}}{\tau - t}\,\overline{d\tau} - 2\operatorname{Re}\Gamma - \frac{dt}{\overline{dt}}\Gamma', \quad t \in L. \tag{21}$$

This is the singular integral equation valid along the crack L. Next, if Eqs. (17) and (20) and the Plemelj formulae [16] are taken into account, the boundary condition (5) on L' can be written as

$$-\{(\kappa_1 + 1) + \Gamma_0(\kappa_2 + 1)\}\overline{\phi(t)} + \frac{1 - \Gamma_0}{\pi i}\int_{L_0} \frac{\phi(\tau)}{\tau - t}\,d\tau + \frac{\kappa_1 - \Gamma_0\kappa_2}{\pi i}\int_{L_0} \frac{\overline{\phi(\tau)}}{\bar{\tau} - \bar{t}}\,\overline{d\tau} - 2(1 - \Gamma_0)\frac{dt}{\overline{dt}}\frac{1}{\pi i}\int_{L_0} \frac{\operatorname{Re}[(\bar{\tau} - \bar{t})\phi(\tau)\,d\tau]}{(\tau - t)^2} = 2G(t) - \frac{dt}{\overline{dt}}\frac{2(1 - \Gamma_0)}{\pi i}\int_L \frac{\overline{q(\tau)}}{\tau - t}\,\overline{d\tau} - 2(1 - \Gamma_0)\Gamma + 2(\kappa_1 - \Gamma_0\kappa_2)\bar{\Gamma} - \frac{dt}{\overline{dt}}(1 - \Gamma_0)\Gamma', \quad t \in L'. \tag{22}$$

This is the singular integral equation valid along the part L' of the interface L_0, where a perfect bonding between the inclusion and the matrix was assumed to exist. In fact, Eqs. (21) and (22) constitute a single complex Cauchy type singular integral equation on the whole boundary of the interface L_0. The closed form, approximate or rather numerical solution of these equations permits the evaluation of the unknown function $\phi(t)$ on the interface L_0. Then the whole problem, as was already mentioned, will have been solved.

During the numerical solution of Eqs. (21) and (22) we have also to take into account the condition of single-valuedness of displacements (11). If we apply the Plemelj formulae to Eqs. (17) and (20), we find that

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i}\int_{L_0}\frac{\phi(\tau)}{\tau - t}\,\mathrm{d}\tau + \Gamma, \tag{23}$$

$$\begin{aligned}\Psi^{\pm}(t) = &\pm\frac{1}{2}\left\{\frac{\overline{\mathrm{d}t}}{\mathrm{d}t}[2\overline{q(t)} - \phi(t) - \overline{\phi(t)}] - \bar{t}\phi'(t)\right\}\\ &+\frac{1}{\pi i}\int_{L}\frac{\overline{g(\tau)}}{\tau - t}\,\overline{\mathrm{d}\tau} - \frac{1}{2\pi i}\int_{L_0}\frac{\overline{\phi(\tau)}}{\tau - t}\,\overline{\mathrm{d}\tau} - \frac{1}{2\pi i}\int_{L_0}\frac{\bar{\tau}\phi(\tau)}{(\tau - t)^2}\,\mathrm{d}\tau + \Gamma'.\end{aligned} \tag{24}$$

By substituting the boundary values of $\Phi(z)$ and $\Psi(z)$ from Eqs. (23) and (24) respectively into Eq. (11), we can express the condition of single-valuedness of displacements in terms of the unknown density function $\phi(t)$ of the integral equation only.

Furthermore, some special cases are of particular interest: In the case when the materials of the matrix and the inclusion are the same ($\mu_1 = \mu_2 = \mu$, $\Gamma_0 = 1$, $\kappa_1 = \kappa_2 = \kappa$), Eq. (22) is simplified as

$$\phi(t) = -\frac{\overline{G(t)}}{\kappa + 1}, \quad t \in L'. \tag{25}$$

In this case, the singular integral equation to be solved is only Eq. (21) extending only on the crack L. Moreover, the condition of single-valuedness of displacements (11) is simplified as

$$(\kappa + 1)\int_{L}\phi(t)\,\mathrm{d}t = 2\int_{L}q(t)\,\mathrm{d}t + 2\mu[g(b) - g(a)]. \tag{26}$$

If, further, $g(t) \equiv 0$ on L' (whence $G(t) \equiv 0$ on L' too), then Eqs. (21) and (26) take exactly the same forms as for a simple curvilinear crack inside an infinite isotropic elastic medium [1]. On the other hand, if the materials of the matrix and the inclusion are different but no crack exists along the interface, then Eqs. (11) and (21) are no more of any use, and Eq. (22) becomes identical to the singular integral equation derived in Ref. [21] for the problem of an inclusion inside an infinite plane isotropic elastic medium.

Finally, the arguments of this section hold also true in the case of a system of n curvilinear cracks L along the interface L_0. Of course, in this case, one has to take into account n conditions of single-valuedness of displacements of the form (11) (with the term $\int_{L'} G(t)\,\overline{\mathrm{d}t}$ always replaced by $2\mu_1[g(b) - g(a)]$), one for each independent crack. It can also be mentioned that up to now no assumption on the shape of the interface was made. This means that the interface may be not smooth, i.e. it may have corner points. In this case, the derived equations remain valid on the whole interface, but with the corner points of it excluded. We can also mention that the generalization of the results of this section to the case of several inclusions (perhaps of different materials) can also be achieved without much difficulty. The same holds also true for the case of an inclusion with an interface crack inside a finite matrix. In this last case, the respective results derived in Ref. [21] for a simple inclusion with no interface crack L inside a finite matrix should be taken into account. Several more generalizations of the present results to cover any practical problem of interface cracks are also possible but of a too trivial character to be mentioned here in detail.

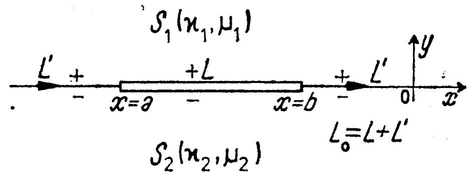

Fig. 2: Geometry of a crack along the straight interface of two isotropic elastic half-planes

4. The case of a straight interface

Now we apply the method developed in the previous sections to the simple case of two bonded elastic half-planes with a crack L along their straight interface as shown in Fig. 2. This problem, as was already mentioned, was treated in Refs. [2–10] and solved in closed form by reduction to a system of Riemann–Hilbert boundary value problems without using Cauchy type singular integral equations. In this section, we want just to verify some of the results of the previous section on the basis of the results of Refs. [2–10]. In this section, we also assume that the existing loading acts only on the crack L and also that $g(t) \equiv 0$ as was also assumed in Refs. [2–10].

In this special case, we consider the Ox-axis of the coordinate system to coincide with the interface of the two half-planes (Fig. 2). Then we have $\bar{\tau} = \tau$ and $\bar{t} = t$ in all the equations derived previously. This fact together with the assumptions that

$$\Gamma = 0, \quad \Gamma' = 0, \quad g(t) \equiv 0, \quad t \in L', \tag{27}$$

causes a simplification of the Cauchy type singular integral equations (21) and (22), which now take the simpler forms

$$\frac{1}{\pi i}\int_{L_0} \frac{\phi(\tau)}{\tau - t}\,\mathrm{d}\tau = p(t) + \frac{1}{\pi i}\int_{L} \frac{q(\tau)}{\tau - t}\,\mathrm{d}\tau, \quad t \in L, \tag{28}$$

and

$$\begin{aligned}\{(\kappa_1+1)+\Gamma_0(\kappa_2+1)\}\phi(t)+\{(\kappa_1-1)-\Gamma_0(\kappa_2-1)\}\frac{1}{\pi i}\int_{L_0}\frac{\phi(\tau)}{\tau-t}\,\mathrm{d}\tau \\ = -\frac{2(1-\Gamma_0)}{\pi i}\int_{L}\frac{q(\tau)}{\tau-t}\,\mathrm{d}\tau, \quad t \in L',\end{aligned} \tag{29}$$

respectively. Moreover, the condition of single-valuedness of displacements (11) is simplified as

$$\begin{aligned}\int_{L}\{[\Phi^{+}(t)-\Gamma_0\Phi^{-}(t)]-[\kappa_1\bar{\Phi}^{+}(t)-\Gamma_0\kappa_2\bar{\Phi}^{-}(t)]+t[\Phi'^{+}(t)-\Gamma_0\Phi'^{-}(t)] \\ +[\Psi^{+}(t)-\Gamma_0\Psi^{-}(t)]\}\,\mathrm{d}t = 0.\end{aligned} \tag{30}$$

The system of singular integral equations (28) and (29) is very simple compared to the system of singular integral equations (21) and (22) and on the basis of the Plemelj formulae (23), it can be reduced to a system of Riemann–Hilbert boundary value problems, that is

$$\Phi^{+}(t)+\Phi^{-}(t) = p(t)+\frac{1}{\pi i}\int_{L}\frac{q(\tau)}{\tau-t}\,\mathrm{d}\tau, \quad t \in L, \tag{31}$$

$$(\kappa_1+\Gamma_0)\Phi^{+}(t)-(1+\Gamma_0\kappa_2)\Phi^{-}(t) = -\frac{1-\Gamma_0}{\pi i}\int_{L}\frac{q(\tau)}{\tau-t}\,\mathrm{d}\tau, \quad t \in L'. \tag{32}$$

Now we will show that the system of Riemann–Hilbert boundary value problems (31) and (32) is equivalent to the system derived in Ref. [8] in a completely different way and solved in closed form. In fact, in Ref. [8], it was shown that the following conditions are valid along L':

$$F_1^+(t) = F_1^-(t), \quad F_2^+(t) = F_2^-(t), \quad t \in L', \tag{33}$$

where

$$F_1(z) = \Phi(z) - \Omega(z), \tag{34}$$

$$F_2(z) = \begin{cases} \kappa_1\mu_2\Phi(z) + \mu_1\Omega(z), & \operatorname{Im} z > 0, \\ \kappa_2\mu_1\Phi(z) + \mu_2\Omega(z), & \operatorname{Im} z < 0, \end{cases} \tag{35}$$

where $\Omega(z)$ is a new complex function [16] related to the complex potentials $\Phi(z)$ and $\Psi(z)$. In Ref. [8], it is further shown that the complex function $F_1(z)$ is determined by

$$F_1(z) = \frac{1}{\pi i}\int_L \frac{q(\tau)}{\tau - z}\,\mathrm{d}\tau. \tag{36}$$

From Eq. (36) it is evident that the first of the conditions (33) is identically satisfied. The second of these conditions with Eqs. (34), (35) and (6) taken also into account can be written as

$$(\kappa_1 + \Gamma_0)\Phi^+(t) - (1 + \Gamma_0\kappa_2)\Phi^-(t) = -[F_1^-(t) - \Gamma_0 F_1^+(t)], \quad t \in L'. \tag{37}$$

Because of the first of Eqs. (33) as well as Eq. (36), Eq. (37) can be shown to be completely equivalent to Eq. (32).

On the other hand, the following boundary condition should be valid on the crack L [8]:

$$[\Phi^+(t) + \Omega^+(t)] + [\Phi^-(t) + \Omega^-(t)] = 2p(t), \quad t \in L. \tag{38}$$

This equation together with Eqs. (34) and (36) leads to the boundary condition (31) derived here by a quite different method.

As regards the condition of single-valuedness of displacements on the crack L, in Ref. [8] it was shown to be equivalent to

$$\int_L [F_2^+(t) - F_2^-(t)]\,\mathrm{d}t = 0. \tag{39}$$

By taking into account Eq. (35) together with Eq. (6) as well as the definition of the complex function $\Omega(z)$ [16]

$$\Omega(z) := \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z), \tag{40}$$

we can easily show that the forms (30) and (39) of the condition of single-valuedness of displacements on the crack L are completely equivalent. Thus, it was shown that the method of singular integral equations used here leads to the same results as the method used in Ref. [8] and based on the reduction of the problem of a crack L along the straight interface of two isotropic elastic half-planes to a system of Riemann–Hilbert boundary value problems. Of course, no such comparison can be made in the general case of a curvilinear interface crack L except if this crack has the shape of a circular or elliptical arc.

5. On the numerical solution of the singular integral equations

In the general case of a curvilinear interface crack (or a system of such cracks), no closed-form solution of the complex singular integral equations (21) and (22) together with the condition (11) is possible. Then the best possibility is to try to solve this system of equations numerically. At this point, we can mention that several effective numerical methods of approximate solution of real or complex singular integral equations are recently available; see, e.g., Refs. [22–24]. These methods have been already successfully applied to the numerical solution of complex singular integral equations arising in crack problems [17, 19] or inclusion problems [21] and being of the same complicated character as Eqs. (21) and (22). Thus, here we will not try to solve these equations numerically restricting ourselves to mentioning one peculiarity of these equations.

This peculiarity consists in the fact that the complex potential $\Phi(z)$ presents complex singularities of orders $(-1/2) \pm i\beta$ at the end-points $t = a$ and $t = b$ of the curvilinear crack (Fig. 1), respectively, where the constant β depends only on the elastic properties of the matrix S_2 and the inclusion S_1 and it is given by [8]

$$\beta = \frac{1}{2\pi} \ln \frac{\kappa_1 \mu_2 + \mu_1}{\kappa_2 \mu_1 + \mu_2} = \frac{1}{2\pi} \ln \frac{\kappa_1 + \Gamma_0}{\kappa_2 \Gamma_0 + 1}\,, \tag{41}$$

where Eq. (6) has also been taken into account. This means that the complex potential $\Phi(z)$ behaves like $z^{(-1/2)\pm i\beta}$ near the points $z = a$ and $z = b$, respectively. Because of Eqs. (23), an analogous behaviour is expected for the unknown function $\phi(t)$ in the singular integral equations (21) and (22) and further for the stress field around the crack tips $z = a$ and $z = b$.

The existence of singularities of complex order near a crack tip on a bimaterial interface is a well-known fact for the first time mentioned by Williams [25] by using the eigenvalue method in its real form. The same fact was established in Ref. [8] by using both the eigenvalue method in its complex form (associated with the complex potentials $\phi(z)$ and $\psi(z)$ of Muskhelishvili [16]), valid both for straight and for curvilinear interface cracks, and the direct method of solution of the problem of a straight interface crack by its reduction to a system of Riemann–Hilbert boundary value problems as was already mentioned. Finally, in a recent note [26], the authors showed that only one complex singularity, $(-1/2) + i\beta$ at $z = a$ and $(-1/2) - i\beta$ at $z = b$, exists at the crack tips and not both these singularities at the same time as was already known for the special case of a straight interface crack. This fact constitutes a great simplification in the numerical solution of the singular integral equations (21) and (22).

For the numerical solution of Eqs. (21) and (22), together with Eq. (11), the method of reduction of these equations to a system of linear equations by approximating the integrals by using an appropriate numerical integration rule and next applying the equations at properly selected collocation points [22–24] is the best possibility. Moreover, in Refs. [23, 24], this method was extended to the case of complex singularities at the end-points of the integration interval (the crack tips in our problem) exactly as in the problem under consideration. The use of the Lobatto–Jacobi numerical integration rule was proposed in these references since this rule permits the direct evaluation of the stress intensity factors at the crack tips. Moreover, in Ref. [23], the Lobatto–Jacobi method was applied to the numerical solution of the singular integral equations for a simple crack or for a periodic array of cracks along the straight boundary of two isotropic elastic half-planes (Fig. 2). This was the first time that this method was used for the numerical solution of singular integral equations with complex singularities. The peculiarity of this technique was that it made use of the Lobatto–Jacobi numerical integration rule with complex abscissae and weights and, moreover, that the collocation points selected lied outside the integration interval. Of course, the same method of numerical solution of singular integral equations can also be successfully applied to the numerical solution of the present singular integral equations (21) and (22) together with the condition (11).

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