Summary

GENERAL METHODS FOR THE SOLUTION OF CRACK PROBLEMS IN THE THEORY OF PLANE ELASTICITY

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1. General Considerations

The problem of an elastic cracked medium under conditions of plane stress or plane strain has not been treated up to now in its general form although the problems of a finite or infinite, simple or composite, simply - connected or multiply-connected medium without cracks have been fully solved. It is the aim of this book to give general and effective methods for the solution of any plane problem of a medium with one or more cracks or sets of cracks.

The solution of such a problem involves the development and solution of the necessary equations. As regards the development of equations, the classical method of complex potentials $\varphi(z)$ and $\psi(z)$, or $\Phi(z)$ and $\Psi(z)$, which has already proved quite effective in solving plane elasticity problems, is used throughout. Integral representations of these potentials are proposed for crack problems in the form of Cauchy-type integrals. Thus the whole problem is to find the unknown densities of these potentials, which for most cases can be interpreted as densities of concentrated forces or dislocation arrays or combinations of them acting on the boundaries of the medium and especially on the cracks. Both the first and the second fundamental problems of the Theory of Plane Elasticity as well as the mixed fundamental problem are considered. Of course, emphasis is put on the solution of the first fundamental problem, which applies to

the usual case of unloaded cracks in a loaded along its other boundaries medium too. By using the Plemelj formulae, the whole problem is reduced in all cases to the solution of one or more Cauchy-type singular integral equations along all boundaries of the medium (the cracks included). These equations are accompanied by the condition or conditions of single-valuedness of displacements, which must always be taken into consideration around each crack or hole inside the medium.

As regards now the solution of the resulting equations, this consists basically of the numerical solution of the Cauchy-type singular integral equations to which a plane crack problem can be reduced, as mentioned above. For the numerical solution of such equations, which have been up to now a problem for all people having to do with them, a quite effective method is introduced which consists in the reduction of these equations into linear systems by applying the well-known numerical quadrature procedures to the Cauchy-type singular integrals. It may be noticed that the use of the methods of numerical integration for Cauchy-type integrals is allowable only when the points of application of the singular integral equations, in which these integrals occur, are determined as roots of trancsendental equations associated with the numerical quadrature procedure in use. In this way, a Cauchy-type singular integral equation can be thought, as to its numerical solution, almost equal in difficulty to a normal integral equation with regular kernels. This method for the numerical solution of singular integral equations can be equally well applied to the solution of such equations ocurring not only in the theory of Plane Elasticity, but also in a lot of branches of Mathematical Physics.

Applications of the above-described method of solution of crack problems are given to a number of cases, most of which have been already solved by other methods, in order that a comparison with the method proposed in this book is possible. It can be seen that for the first time the values of the stress intensity factors at the tips of the cracks are found directly from the solution of the singular integral equations without the need of using the methods of extrapolation or of infinite sums.

The development of general computer programs for the solution of any crack problem that might occur in a possible application and the tabulation of the points of application of a Cauchy-type singular integral equation are beyond the scope of this book but, it is hoped, will be treated in future works. It is thought that the above-mentioned points could be a complement to the already existing tables of abscissas and weights of the numerical integration methods.

2. Abstracts

The whole book is divided into four chapters, each of which is further subdivided into nine to ten sections. Because of the fact that different problems are solved in these sections, short abstracts of all sections of the book are given below, which, we believe, will prove useful to a reader wishing to study only some problems among those treated in this book.

It must also be noticed that, when speaking about singular integrals and singular integral equations, we mean Cauchy-type singular integrals and singular integral equations with Cauchy-type kernels respectively, and also that the terms Gaussian quadrature method, Radau quadrature method and Lobatto quadrature method are used to denote the Gauss-type quadrature method with all its abscissas inside the integration interval, with one of its abscissas coinciding with one of the ends of the integration interval and with two of its abscissas coinciding with both ends of the integration interval respectively.

CHAPTER A′: **The problem of the simple smooth crack in an infinite medium**

In this chapter the problem of the simple smooth crack in an infinite isotropic or anisotropic medium is considered. For the case of an isotropic medium, the first fundamental, the second fundamental and the mixed fundamental problem are studied, while for the case of an anisotropic medium only the first fundamental problem is examined. Several methods of solution for the first fundamental problem are proposed as it is the most important. Throughout the chapter the method of complex potentials is used and the problems considered are reduced to complex singular integral equations along the crack together with a condition of single-valuedness of displacements.

A1. General method of solution of the problem of the simple smooth crack in an infinite isotropic medium: The first fundamental problem for an infinite medium containing a simple smooth crack is reduced to a complex singular integral equation along the crack and a condition of single-valuedness of displacements, by using the method of complex potentials $\Phi(z)$ and $\Psi(z)$ together with the Plemelj formulae. The unknown function in the singular integral equation, which is the density of a Cauchy-type integral representing the function $\Phi(z)$, can be related to the densities of concentrated forces and dislocations along the crack. These densities could be called the concentrated forces function and the dislocation function respectively. Reduction of the complex singular integral equation to two real singular integral equations and application to the problem of a straight crack are also given.

A2. Another method of solution of the problem of the simple smooth crack in an infinite isotropic medium: The same technique described in section A1 is used to reduce the first fundamental problem for a simple smooth crack in an infinite isotropic medium to a complex singular integral equation together with the condition of single-valuedness of displacements. The only difference consists in a different expression of the complex function $\Phi(z)$ through a Cauchy-type integral. Two other Fredholm-type integral equations equivalent to the singular integral equations of section A1 and the present section are also derived, but their form is rather complicated.

A3. Solution of the problem of the simple smooth crack in an infinite isotropic medium by determination of complex potentials $\varphi_o(z)$ and $\psi_o(z)$: One more method for the solution of the first fundamental problem for a simple smooth crack in an infinite isotropic medium can be derived by using the complex potentials $\varphi_o(z)$ and $\psi_o(z)$ instead of $\Phi(z)$ and $\Psi(z)$ used in section A1. The complex singular integral equation thus derived is proved to be equivalent to that found in section A1.

A4. Solution of the problem of the simple smooth crack in an infinite isotropic medium by reduction to the problem of a body without cracks: An alternative method of solution of the first fundamental problem for a simple smooth crack in an infinite isotropic medium consists in considering the crack as a part of the boundary of two isotropic media, one of which is finite and the other infinite, in which the whole isotropic plane can be divided. The method of complex potentials for the solution of non-cracked media can be used for the problem considered and, after a proper modification, can reduce it to a complex singular integral equation equivalent to that of section A3.

A5. Solution of the problem of the simple smooth crack in an infinite isotropic medium by using its solution for a simple straight crack: One last method of treating the first fundamental problem for a simple smooth crack in an infinite isotropic medium consists in properly modifying its classical solution for the special case of a straight crack. The problem can be reduced to a system of two singular integral equations along the crack together with the necessary condition of single-valuedness of displacements.

A6. A general method of solution of the second fundamental problem for a simple smooth crack in an infinite isotropic medium: The second fundamental problem for a simple smooth crack in an infinite isotropic medium can be treated in a quite analogous way to that used in sections A1 to A5 for the first fundamental problem. Here the method of section A1 is used to reduce this problem to a singular integral equation along the crack together with the condition of the resultant force applied on the crack.

A7. A general method of solution of the mixed fundamental problem for a simple smooth crack in an infinite isotropic medium: The mixed fundamental problem for a simple smooth crack in an infinite isotropic medium can be reduced to a complex singular integral equation along the crack together with the condition of single-valuedness of displacements. This equation depends on the existing boundary conditions on the edges of the crack.

A8. A general method of solution of the problem of a simple smooth crack in an infinite anisotropic medium: The first fundamental problem for an infinite anisotropic medium containing a simple smooth crack is reduced to a complex singular integral equation along the crack together with a condition of single-valuedness of displacements, in a similar way to that used in section A1 for the isotropic case except for the use of complex potentials $\Phi(z_1)$ and $\Psi(z_2)$ instead of $\Phi(z)$ and $\Psi(z)$.

A9. Another method of solution of the problem of the simple smooth crack in an infinite anisotropic medium: The same method used in section A2 for the isotropic case can, properly modified, be used for the solution of the first fundamental problem for a simple smooth crack in an infinite anisotropic medium. The problem is reduced to a complex singular integral equation equivalent to that derived in section A8. Two other equivalent Fredholm-type integral equations are also given. All these equations are accompanied by the condition of single-valuedness of displacements.

CHAPTER B′: **Complex crack problems**

In this chapter more complex crack problems are considered, as cracks in finite or composite media or arrays of cracks. Only the case of the first fundamental problem and of isotropic media is considered because of the fact that effective methods of solution can be easily derived for the cases of the other fundamental problems as well as of anisotropic media if the corresponding methods of solution derived in Chapter A′ for the simple smooth crack are taken into consideration. Also, useful formulas for the calculation of stress intensity factors are given. The method of complex potentials is used throughout the chapter.

B1. Cracks in an infinite isotropic medium: The first fundamental problem for an infinite medium with cracks is reduced to a complex singular integral equation along the cracks together with conditions of single-valuedness of displacements equal in number to the number of cracks. Application to the case of straight cracks is given in more detail.

B2. Crack in a finite isotropic medium: The first fundamental problem for a finite isotropic medium containing a crack is reduced to a complex singular integral equation along both the crack and the boundary of the medium together with a condition of single-valuedness of displacements around the crack.

B3. Crack between two isotropic media: The first fundamental problem for an infinite isotropic medium containing an isotropic inclusion with a crack along a part of its boundary is reduced to a complex singular integral equation along the entire boundary of the inclusion, the crack included, together with a condition of single-valuedness of displacements around the crack.

B4. Row of periodic cracks in an infinite isotropic medium: The first fundamental problem for a row of periodic smooth cracks in an infinite isotropic medium is reduced to a complex singular integral equation along one of the cracks together with a condition of single-valuedness of displacements. The derivation of this equation is based either on the theory of periodic functions or on the theory given in section B1 and regarding the interaction between several cracks in an infinite isotropic medium. The method of conformal mapping can be also used.

B5. Doubly-periodic array of cracks in an infinite isotropic medium: The first fundamental problem for a doubly-periodic array of smooth cracks in an infinite isotropic medium is reduced to a complex singular integral equation along one of the cracks together with a condition of single-valuedness of displacements. The derivation of this equation is based on the theory of doubly-periodic functions.

B6. Array of radial cracks in an infinite isotropic medium: The first fundamental problem for an array of radial smooth cracks in an infinite isotropic medium is reduced to a complex singular integral equation along one of the cracks together with a condition of single-valuedness of displacements. The derivation of this equation is based either on the theory of automorphic functions or on the theory given in section B1 and regarding the interaction of several cracks in an infinite isotropic medium. The method of conformal mapping can also be used. Application to the case of an array of straight radial cracks is also given.

B7. Crack with angular points or sets of cracks with a common point in an infinite isotropic medium: The problem of a crack with angular points or a set of cracks with a common point is studied with emphasis on the problem of finding the singularities of the complex potentials around angular points or common points of cracks. Afterwards, the reduction of the problem to the problem of a smooth crack studied in section A1 is easily possible.

B8. Case of anisotropic media - Generalizations: The methods of treating crack problems studied in this chapter for the case of the first fundamental problem and of isotropic media can be properly modified so as to apply to the cases of the second or the mixed fundamental problems as well as the case of anisotropic media. It is also possible to combine the above methods so that more complex crack problems, than that considered in this chapter, can be solved.

B9. Stress intensity factors: Useful formulas are given for the calculation of stress intensity factors at the tips of cracks, supposing that the complex potential $\Phi(z)$ or the dislocation function or some other equivalent function has been previously calculated. An example for the case of an arc-shaped crack is also given.

CHAPTER Γ′: Numerical solution of singular integral equations

In this chapter an effective method for the numerical solution of singular integral equations with Cauchy-type kernels, as those derived in chapters A′ and B′, is presented. By using this method, a singular integral equation can be numerically solved by about the same amount of computation to that required for the numerical solution of an integral equation with regular kernels.

Γ1. Introduction: The methods of numerical quadrature as applied to the solution of integral equations are presented in a general way.

Γ2. On the existing methods of numerical solution of singular integral equations: A brief account of the already existing methods of numerical solution of singular integral equations is given accompanied by the advantages and disadvantages of each one of them.

Γ3. The Gauss-Chebyshev and Lobatto-Chebyshev methods for the numerical solution of singular integral equations: The Gauss-Chebyshev method for the numerical solution of singular integral equations is proved to be, in reality, a Gaussian integration method by consideration of its accuracy, which was not thoroughly investigated previously. The method Lobatto-Chebyshev for the numerical solution of singular integral equations is further derived for the first time and in a way similar to that used for the already known Gauss-Chebyshev method. These methods basically make use of the corresponding methods of numerical quadrature, but not in an explicit way, for the calculation of the Cauchy-type integrals of the singular integral equations and find proper points of application of these equations so that the Cauchy-type singularities do not influence the accuracy of calculation of the Cauchy-type integrals. The advantage of the Lobatto-Chebyshev method over the Gauss-Chebyshev method, in case of crack problems, is that by the use of the Lobatto-Chebyshev method the values of the stress intensity factors at the tips of the cracks can be directly determined after the numerical solution of the corresponding singular integral equations.

Γ4. Application of the Gauss quadrature method to the calculation of singular integrals: The Gauss quadrature method can be properly modified so as to apply to the numerical calculation of Cauchy-type singular integrals. This modification consists in including one more term in the approximate expression of such an integral. The accuracy of this modified Gauss quadrature method is quite equal to the accuracy of the usual Gauss quadrature method.

Γ5. Application of a general method of numerical integration to the calculation of singular integrals: A Cauchy-type singular integral can be computed numerically in the same way as an ordinary integral, except for the fact that the simple pole of the integrated function inside the integration interval must be taken into account. In this way, a new term, not existing in the approximate expressions of ordinary integrals, should be added. By this method, any quadrature formula used for ordinary integrals can be modified to be used for Cauchy-type singular integrals too.

Γ6. Application of the Radau and Lobatto quadrature methods to the calculation of singular integrals: The Radau and Lobatto methods of numerical quadrature are derived as special cases of the general method of calculation of singular integrals developed in section Γ5.

Γ7. Approximate method of solution of singular integral equations: A singular integral equation can be numerically solved if its integrals are expressed in an approximate way by the use of a proper quadrature formula and then the equation is applied to certain points inside the integration interval. In this way, a system of linear equations results, which can be easily solved. The points of application of the singular integral equation must be selected as the the roots of certain transcendental functions in such a way that the extra term in the approximate expression of a singular integral vanishes when this integral is numerically expressed as an ordinary integral.

Γ8. Calculation of a function in an interval by means of its values at certain points of the interval by interpolation: Brief accounts of the well-known methods of interpolation are given as applied to cases when a function has been determined only at the abscissas of a numerical quadrature rule and need to be determined over all the corresponding integration interval. This case results after the solution of the system of linear equations approximating a singular integral equation has been found.

Γ9. Common cases of singular integral equations and methods for their solution: The numerical method of solution of singular integral equations described in section Γ7 is applied to the most usual cases of numerical integration rules used for the calculation of the integrals of the singular integral equations. For each one of these rules, which are of the Gauss, Radau and Lobatto type, the functions, the roots of which are the points of application of the corresponding singular integral equations, are given in an explicit form. These functions are derived from the orthogonal polynomials associated with the considered quadrature rule. In this way, each rule applicable to a certain integration interval and a weight function and further to a singular integral equation containing a Cauchy-type integral, with the same integration interval and weight function, can be thought, when complemented by the values of the points of application of the singular integral equation, as a rule for the numerical solution of the singular integral equation. Thus, the following methods for the solution of singular integral equations have been considered: Gauss-Legendre, Lobatto-Legendre, Gauss-Chebyshev, Lobatto-Chebyshev, Gauss-Jacobi, Gauss-Laguerre and Gauss-Hermite, as well as modified forms of the Gauss-Legendre, Lobatto-Legendre and Gauss-Hermite methods through the change of variable: $\tau = t^2$ or $\tau = 1-t^2$, where t is the variable of integration for the normal form of the rule and τ for its modified form.

These modified rules are equivalent to Gauss, Radau and Lobatto rules and very useful for the numerical solution of singular integral equations ocurring in crack problems. Tables of the abscissas of application of a singular integral equation for its numerical solution are given in the cases of the Gauss-Legendre, modified Gauss-Legendre, Lobatto-Legendre and modified Lobatto-Legendre methods.

Γ10. Remarks: Useful remarks regarding the accuracy of a method of numerical quadrature, when applied to the solution of a singular integral equation, are given.

CHAPTER Δ′: Applications to definite crack problems

In this chapter the methods of solution of crack problems developed in chapters A′ and B′ are applied to some known crack problems. For the solution of the resulting singular integral equations, the Radau and Lobatto methods have been preferred over the corresponding Gauss methods because of the fact that in this way the stress intensity factors at the tips of the cracks can be directly computed. Comparison of these methods of treating crack problems with other already existing methods is made wherever possible. Tables of the values of the stress intensity factors at the tips of the cracks are also given. Examples for the cases of regularly distributed, intersecting or branched cracks are considered.

Δ1. Application 1: Row of collinear periodic cracks: The well-known problem of a row of collinear periodic cracks in an infinite isotropic medium is considered. All cracks are loaded with a constant pressure. The problem is reduced to a singular integral equation along one of the cracks, which is solved by the Lobatto-Chebyshev method. The distribution of the dislocation function along the cracks is given for some typical cases as well as the corresponding values of the stress intensity factor at the tips of the cracks, which are in agreement with its theoretical values found by the closed-form solution of the present problem. Extension of the method of solution to more complicated cases of rows of periodic cracks is quite possible.

Δ2. Application 2: Row of parallel periodic cracks: The problem of a row of parallel periodic cracks in an infinite isotropic medium and with a constant pressure along the edges of the cracks is considered. The problem is reduced to a singular integral equation along one of the cracks, which is solved by the Lobatto-Chebyshev method as in the case of Application 1. The distribution of the dislocation function along the cracks is given for some typical cases as well as the corresponding values of the stress intensity factors at the tips of the cracks, which are in agreement with their values already found by other methods. Extension of the method of solution to more complicated cases of rows of periodic cracks is quite possible.

Δ3. Application 3: Doubly-periodic array of cracks: The problem of a doubly-periodic array of cracks in an infinite isotropic medium and with a constant pressure along the edges of the cracks is considered. The problem is reduced to a singular integral equation along one of the cracks , which is solved by the Lobatto-Chebyshev method as in the cases of Applications 1 and 2. The distribution of the dislocation function along the cracks is given for some typical cases as well as the corresponding values of the stress intensity factors at the tips of the cracks, which are in agreement with their values already found by other methods. Extension of the method of solution to more complicated cases of doubly-periodic arrays of cracks is quite possible.

Δ4. Application 4: Symmetrical star-shaped crack: The problem of a star-shaped crack in an infinite isotropic medium and with a constant pressure along the edges of the crack is considered. The problem is reduced to a singular integral equation along one of the cracks, which is solved by the modified Gauss-Legendre method. The distribution of the dislocation function along the crack is given for various numbers of the branches of the star-shaped crack as well as the corresponding values of the stress intensity factors at the tips of the cracks, which are in agreement with their values already found by other methods. Extension of the method of solution to more complicated cases of cracks with a radial symmetry is quite possible.

Δ5. Application 5: Cruciform crack: The problem of a cruciform crack in an infinite isotropic medium and with a constant pressure along the edges of the crack is considered. This problem, in the case of an unsymmetrical cruciform crack, is reduced to a system of singular integral equations along the two arms of the crack, which is simplified to one singular integral equation in the case of a symmetrical cruciform crack. These equations are solved by the modified Gauss-Legendre, the Gauss-Chebyshev and the Lobatto-Chebyshev methods. The distribution of the dislocation function along the crack is given for various ratios of the lengths of the arms of the crack as well as the corresponding values of the stress intensity factors at the tips of the crack , which are in agreement with their values already found by other methods. Extension of the method of solution to more complicated cases of intersecting cracks is quite possible.

Δ6. Application 6: Edge crack in a half-plane: The problem of an edge crack normal to the boundary of an isotropic half-plane and with a constant loading at infinity and normal to the crack is considered. The problem is reduced to a system of two singular integral equations along the crack and the boundary of the half-plane, which is solved by the use of both the modified Gauss-Legendre and the Gauss-Laguerre methods. The distribution of the dislocation function along the boundary of the half-plane and the crack is given as well as the value of the stress intensity factor at the tip of the crack, which is in agreement with its value already found by other methods. Extension of the method of solution to more complicated cases of edge cracks is quite possible.

Δ7. Application 7: Simple smooth crack in an infinite isotropic medium: The problem of a simple smooth crack in an infinite isotropic medium and with a constant loading at infinity is considered. The system of two singular integral equations derived for this problem in section A1 is solved for the cases of a straight crack as well as an arc-shaped crack by the use of the Lobatto-Chebyshev method. The distribution of the dislocation functions along the cracks is given as well as the values of the stress intensity factors at the tips of the cracks, which are in agreement with their theoretical values. Application of the method of solution to any case of a simple smooth crack in an infinite isotropic medium is quite possible.

Δ8. Application 8: Branched crack: The problem of a branched crack in an infinite isotropic medium and with a constant loading at infinity and normal to the branched crack is considered. The problem is reduced to a system of three singular integral equations along the composite crack, which are solved either by the use of both the modified Gauss-Legendre and the modified Lobatto-Legendre methods or by the use of only the modified Lobatto-Legendre method together with two obvious conditions at the point of branching of the main crack. The resulting values of the stress intensity factors at the three tips of the composite crack are given and are in satisfactory agreement with their experimental values since no theoretical values have been derived for this problem. Extension of the method of solution to any problem of cracks emanating from a common point is quite possible.

Δ9. Application 9: Row of parallel semi-infinite periodic cracks: The problem of a row of parallel semi-infinite periodic cracks in an infinite isotropic medium and with a constant pressure applied along the edges of the cracks is considered. The problem is reduced to a singular integral equation along one of the cracks, which is solved by the modified Gauss-Hermite method. The distribution of the dislocation function along the cracks is given as well as the value of the stress intensity factor at the tips of the cracks, which is in agreement with its value already found by other methods. Extension of the method of solution to more complicated cases of rows of semi-infinite periodic cracks is quite possible.

ΕΙΣΑΓΩΓΗ

1. ΓΕΝΙΚΟΤΗΤΕΣ

Ἡ Θεωρία τῆς Ἐλαστικότητος, μέ ἐφαρμογάς τῆς ὁποίας εἰς προβλήματα σωμάτων μετά ρωγμῶν ἀσχολούμεθα εἰς τήν παροῦσαν ἐργασίαν, ἔχει ἱστορίαν ἑνός περίπου αἰῶνος, ἀλλ' ἡ ἔντονος ἀνάπτυξίς της ἔλαβε χώραν κατά τά τελευταῖα τεσσαράκοντα ἔτη, κατά τήν διάρκειαν τῶν ὁποίων ἐπελύθησαν πλεῖστα προβλήματα τόσον θεωρητικοῦ ὅσον καί πρακτικοῦ ἐνδιαφέροντος, μεταξύ τῶν ὁποίων καί προβλήματα σωμάτων μετά ρωγμῶν.

Τά προβλήματα, μέ τά ὁποῖα θά ἀσχοληθῶμεν εἰς τήν παροῦσαν ἐργασίαν καί τῶν ὁποίων θά προσπαθήσωμεν νά εὕρωμεν τάς λύσεις ἤ τοὐλάχιστον νά ὑποδείξωμεν γενικάς μεθόδους ἀντιμετωπίσεως, ἀφοροῦν εἰς τήν εὕρεσιν τῶν πεδίων τῶν τάσεων καί τῶν μετατοπίσεων ἐντός πεπερασμένων ἤ ἀπείρων σωμάτων μετά ρωγμῶν ἀλλ' εὑρισκομένων εἰς κατάστασιν ἐπιπέδου καταπονήσεως. Τοῦτο σημαίνει ὅτι τά σώματα ταῦτα εἶναι εἴτε ἀπείρως μικροῦ πάχους ἐκτεινόμενα μόνον κατά τάς δύο διαστάσεις, ὅτε ἔχομεν τήν περίπτωσιν τῆς ἐπιπέδου ἐντατικῆς καταστάσεως, εἴτε ἀπείρως μεγάλου πάχους ἐκτεινόμενα καί κατά τάς τρεῖς διαστάσεις, ἀλλά μέ ὅρια μεταβαλλόμενα μόνον κατά τάς δύο διαστάσεις καί ἀνεξάρτητα τῆς τρίτης διαστάσεως, ὅτε ἔχομεν τήν περίπτωσιν τῆς ἐπιπέδου παραμορφωσιακῆς καταστάσεως. Πάντως ἡ διαπραγμάτευσις ἀμφοτέρων τῶν περιπτώσεων τούτων εἶναι ὁμοία.

Περαιτέρω δεχόμεθα ὅτι ἐπί τῶν τυχόν ὑπαρχόντων ὁρίων τῶν θεωρουμένων σωμάτων ὡς καί ἐπ' ἀμφοτέρων τῶν πλευρῶν τῶν ρωγμῶν δίδονται ἤ αἱ συνιστῶσαι τῶν ἐξωτερικῶς ἐπιβαλλομένων τάσεων, κάθετος καί διατμητική, ὅτε λέγομεν ὅτι ἔχομεν τήν περίπτωσιν τοῦ πρώτου θεμελιώδους προβλήματος, ἤ αἱ ὑφιστάμεναι μετατοπίσεις, ὅτε λέγομεν ὅτι ἔχομεν τήν περίπτωσιν τοῦ δευτέρου θεμελιώδους προβλήματος, ἤ τέλος ἐπί τινων μέν ὁρίων τοῦ σώματος αἱ συνιστῶσαι τῶν ἐξωτερικῶς ἐπιβαλλομένων τάσεων, ἐπί δέ τῶν ὑπολοίπων ὁρίων τοῦ σώματος αἱ ὑφιστάμεναι μετατοπίσεις, ὅτε λέγομεν ὅτι ἔχομεν τήν περίπτωσιν τοῦ μικτοῦ θεμελιώδους προβλήματος. Δυνάμεθα βεβαίως νά θεωρήσωμεν καί ἄλλων τύπων ὁριακάς συνθήκας, τά ἀντίστοιχα ὅμως προβλήματα εἶναι ὀλιγώτερον συνήθη τῶν τριῶν προηγουμένως ἀναφερθέντων.

Διά τήν ἐπίλυσιν τῶν προαναφερθέντων προβλημάτων θά χρησιμοποιήσωμεν τήν Θεωρίαν τῆς Ἐλαστικότητος, ἥτις ἔχει ὡς βάσιν τόν νόμον τοῦ HOOKE, κατά τόν ὁποῖον αἱ συνιστῶσαι τῶν τάσεων καί αἱ συνιστῶσαι τῶν παραμορφώσεων, αἵτινες ὑφίστανται ἐντός ἑνός μέσου, συνδέονται διά γραμμικῶν σχέσεων. Ἡ Θεωρία τῆς Ἐλαστικότητος δίδει σημαντικῶς ἡπλοποιημένους τύπους, ὅταν ἐφαρμοσθῇ εἰς ἐπίπεδα προβλήματα, ὡς τά ἐνταῦθα ἐξετασθησόμενα.

Διά τήν ἐπίλυσιν τῶν προβλημάτων τῆς ἐπιπέδου Ἐλαστικότητος δύνανται νά χρησιμοποιηθοῦν διάφοροι μέθοδοι. Ἐκ τούτων ἡ πλέον κοινή, τήν ὁποίαν καί ἀποκλειστικῶς κατωτέρω θά χρησιμοποιήσωμεν, εἶναι ἡ βασιζομένη ἐπί τῆς θεωρίας τῶν μιγαδικῶν συναρτήσεων καί ἐπιδιώκουσα τόν προσδιορισμόν δύο μιγαδικῶν δυναμικῶν, συναρτήσεων τῆς θέσεως ἐντός τοῦ ἑκάστοτε θεωρουμένου σώματος καί μεταβαλλομένων βεβαίως μόνον κατά τάς δύο διαστάσεις, καθ᾽ ὅσον τά ἐξεταζόμενα ἐνταῦθα προβλήματα ἐθεωρήθησαν ἐπίπεδα.

Δύναται νά σημειωθῇ ὡσαύτως ὅτι διά τῆς προαναφερθείσης μεθόδου τῶν μιγαδικῶν δυναμικῶν ἔχει ληφθῇ τό μεγαλύτερον μέρος τῶν ἀποτελεσμάτων τῆς Θεωρίας τῆς ἐπιπέδου Ἐλαστικότητος. Ἕτεραι διαδεδομέναι μέθοδοι ἐπιλύσεως προβλημάτων τῆς Θεωρίας τῆς ἐπιπέδου Ἐλαστικότητος εἶναι ἡ μέθοδος τῶν ὁλοκληρωτικῶν μετασχηματισμῶν, ὡς ὁ μετασχηματισμός FOURIER καί ὁ μετασχηματισμός MELLIN, καί ἡ μέθοδος τῶν πεπερασμένων στοιχείων. Ἐκ τούτων ἡ πρώτη παρουσιάζει τά μειονεκτήματα ὅτι εἰς μικρόν μόνον ἀριθμόν προβλημάτων δύναται νά ἐφαρμοσθῇ καί ὅτι εἶναι πολύπλοκος, ἐνῷ ἡ δευτέρα παρουσιάζει τό μειονέκτημα ὅτι εἶναι ἐντελῶς ἀριθμητική μέθοδος καί μικρᾶς ἀκριβείας.

2. ΓΕΝΙΚΑΙ ΠΑΡΑΠΟΜΠΑΙ

Πέραν τῶν παραπομπῶν τῶν διδομένων εἰς ἕκαστον τμῆμα τῆς παρούσης μελέτης καί ἀφορωσῶν εἰς τά ἐπί μέρους ἐξεταζόμενα θέματα, δυνάμεθα νά ἀναφέρωμεν καί ὡρισμένας γενικοῦ χαρακτῆρος ἐργασίας, τό περιεχόμενον τῶν ὁποίων ἐλήφθη ὑπ' ὄψιν κατά τήν συγγραφήν τῆς παρούσης μελέτης.

Οὕτω, μεταξύ τῶν γενικοῦ ἐνδιαφέροντος συγγραμμάτων τῶν ἀναφερομένων εἰς τήν θεωρίαν τῆς ἐλαστικότητος δυνάμεθα νά ἀναφέρωμεν τά τῶν TIMOSHENKO and GOODIER {1970}, LANDAU and LIFSHITZ {1970}, SOLOMON {1968}, ΘΕΟΧΑΡΗ {1970} καί LITTLE {1973}. Πλείονας παραπομπάς εὑρίσκομεν εἰς τό ἐνδιαφέρον ἄρθρον τοῦ TEODORESCU {1964}. Περαιτέρω, ἡ ἐνταῦθα χρησιμοποιουμένη μέθοδος τῶν μιγαδικῶν δυναμικῶν ἀναπτύσσεται εἰς τά συγγράμματα τῶν MUSKHELISHVILI {1953A}, GREEN and ZERNA {1968}, MILNE-THOMSON {1968} καί ENGLAND {1971A} ὡς καί εἰς τά ἄρθρα τῶν MUSCHELISVILI {1933}, STEVENSON {1945} καί TIFFEN {1952}. Εἰδικώτερον, ἡ θεωρία τῆς ἐπιπέδου ἐλαστικότητος διά τήν περίπτωσιν ἀνισοτρόπων μέσων ἀναπτύσσεται εἰς τά συγγράμματα τῶν LEKHNITSKII {1963,1968}, SAVIN {1961} καί ΓΑΛΙΔΑΚΗ {1968}.

Ἐφαρμογάς τῆς θεωρίας τῆς ἐπιπέδου ἐλαστικότητος εἰς προβλήματα ρωγμῶν εὑρίσκομεν ἐπίσης εἰς τό σύγγραμμα τῶν SNEDDON and LOWENGRUB {1969}, τό ἄρθρον τοῦ HAHN {1970}, τήν συλλογήν ἄρθρων τοῦ SIH {1973}, τήν ἐργασίαν τοῦ ΙΩΑΚΕΙΜΙΔΗ {1973} καί τό ἐγχειρίδιον τῶν TADA, PARIS and IRWIN {1973}, εἰς τό ὁποῖον ὑπάρχει καί πλήρης σειρά παραπομπῶν σχετικῶν μέ τόν προσδιορισμόν τῶν συντελεστῶν ἐντάσεως τῶν τάσεων εἰς τά ἄκρα ρωγμῶν, οἵτινες δέ πρέπει νά συγχέωνται μέ τούς συντελεστάς συγκεντρώσεως τάσεων εἰς μέσα ἄνευ ρωγμῶν, περί τῶν ὁποίων ἀναφέρουν οἱ NEUBER and HAHN {1966} καί ὁ PETERSON {1974}.

Διά τόν πειραματικόν προσδιορισμόν τῶν συντελεστῶν ἐντάσεως τῶν τάσεων εἰς τά ἄκρα ρωγμῶν ἡ πλέον ἐπιτυχῶς χρησιμοποιουμένη μέθοδος εἶναι ἡ μέθοδος τῶν καυστικῶν ἡ ἀναπτυχθεῖσα ὑπό τοῦ THEOCARIS {1970} καί ἐφαρμοσθεῖσα εἰς πλεῖστα προβλήματα ρωγμῶν ὑπό τοῦ THEOCARIS, σύνοψιν τῶν ὁποίων εὑρίσκομεν εἰς τό ἄρθρον τοῦ THEOCARIS {1972}, ὡς καί ὑπό τῶν THEOCARIS and JOAKIMIDES {1971} καί ΓΔΟΥΤΟΥ {1973}.

Περαιτέρω, μεταξύ τῶν ἀναφερουσῶν γενικάς μεθόδους ἐπιλύσεως τῶν προβλημάτων τῆς ἐπιπέδου ἐλαστικότητος ἐργασιῶν δυνάμεθα νά ἀναφέρωμεν τά ἄρθρα τῶν LAURICELLA {1909}, BESKIN {1944}, RAUSCH {1966}, FINE and NILSON {1966}, RIZZO {1967}, OLIVEIRA {1968}, SEGEDIN and BRICKELL {1968}, BENJUMEA and SIKARSKIE {1972} καί BOWIE, FREESE and NEAL {1973}, εἰς τά ὁποῖα ὅμως δέν ἀντιμετωπίζεται κατά βάσιν τό πρόβλημα τοῦ μετά ρωγμῶν μέσου.

Ὅσον ἀφορᾷ τέλος εἰς τό μαθηματικόν μέρος τῆς παρούσης ἐργασίας, ἐγένετο χρῆσις τῶν συγγραμμάτων τῶν AHLFORS {1966}, PHILLIPS {1957,1966} καί NEVANLINNA and PAATERO {1969} ἐπί τῶν μιγαδικῶν συναρτήσεων, τῶν συγγραμμάτων τῶν MUSKHELISHVILI {1953B}, TRICOMI {1957}, MIKHLIN {1957}, GAKHOV {1966}, POGORZELSKI {1966} καί DELVES and WALSH {1974} ὡς καί τοῦ ἄρθρου τοῦ WOODS {1971} ἐπί τῶν ὁλοκληρωμάτων CAUCHY καί τῶν ὁλοκληρωτικῶν ἐξισώσεων, τῶν συγγραμμάτων τῶν HASTINGS {1955}, RAINVILLE {1960}, ABRAMOWITZ and STEGUN {1965} καί BELL {1968} ἐπί διαφόρων παρουσιαζομένων εἰδικῶν συναρτήσεων, τοῦ συγγράμματος τοῦ FORD {1957} ἐπί αὐτομόρφων συναρτήσεων, τῶν συγγραμμάτων τῶν MINEUR {1952}, HILDEBRAND {1956}, KOPAL {1961} καί RALSTON {1965} ἐπί θεμάτων ἀριθμητικῆς ἀναλύσεως καί εἰδικώτερον ἀριθμητικῆς ὁλοκληρώσεως, εἰς τήν ὁποίαν ἀποκλειστικῶς ἀναφέρονται τά συγγράμματα τῶν STROUD and DON SECREST {1966} καί DAVIS and RABINOWITZ {1967}, καί τῶν συγγραμμάτων τῶν SZEGÖ {1959} καί TRICOMI {1961} ἐπί ὀρθογωνίων πολυωνύμων.

3. ΕΠΙ ΤΩΝ ΟΛΟΚΛΗΡΩΜΑΤΩΝ ΤΥΠΟΥ CAUCHY

Εὑρεῖα χρῆσις γίνεται εἰς τήν παροῦσαν μελέτην τῶν ὁλοκληρωμάτων τύπου CAUCHY τῆς μορφῆς:

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau \,, \tag{1}$$

ἔνθα L κλειστή ἤ ἀνοικτή καμπύλη ἐπί τοῦ μιγαδικοῦ ἐπιπέδου z.

Ἐάν τό σημεῖον z δέν ἀνήκῃ εἰς τήν καμπύλην L, ἐπαρκής συνθήκη ὑπάρξεως τοῦ ὁλοκληρώματος (1) εἶναι νά εἶναι ἡ συνάρτησις $\varphi(t)$ τῶν σημείων t τῆς καμπύλης L ὁλοκληρώσιμος κατά τήν ἔννοιαν τοῦ RIEMANN ὡς πρός t ἐντός παντός τμήματος τῆς καμπύλης L μή περιλαμβάνοντος σημεῖόν τι ταύτης c_k $(k = 1,2,\ldots,m)$, παρά τό ὁποῖον ἡ συνάρτησις $\varphi(t)$ παρουσιάζει ἀσθενῆ ἰδιομορφίαν τῆς μορφῆς {MUSKHELISHVILI , 1953B, §10}:

$$|\varphi(t)| \leq \frac{C_k}{|t - c_k|^{\alpha_k}} \,, \quad C_k > 0 \,, \quad 0 < \alpha_k < 1 \,, \quad k = 1,2,\ldots,m \,. \tag{2}$$

Ὁ ἀριθμός m τῶν σημείων c_k πρέπει ὁπωσδήποτε νά εἶναι πεπερασμένος.

Ἐάν ἀντιθέτως τό σημεῖον z συμπίπτῃ μέ ἕν τῶν σημείων t τῆς καμπύλης L πλήν τυχόν ὑπαρχόντων ἄκρων αὐτῆς, τό ὁλοκλήρωμα (1) ἐν γένει δέν ἔχει ἔννοιαν. Εἰς τήν περίπτωσιν ταύτην ὁρίζομεν τήν συνάρτησιν $\Phi(t)$ ὡς {MUSKHELISHVILI, 1953B, §12}:

$$\Phi(t) = \frac{1}{2\pi i}\lim_{\varepsilon \to 0}\int_{L-\ell} \frac{\varphi(\tau)}{\tau - t}\,d\tau \,, \tag{3}$$

ἔνθα ℓ εἶναι τό τμῆμα τῆς καμπύλης L τό περιεχόμενον ἐντός μικροῦ κύκλου κέντρου t καί ἀκτῖνος ε θεωρουμένης τεινούσης εἰς τό μηδέν. Ἱκανή συνθήκη ὑπάρξεως τοῦ ὁλοκληρώματος (3), τό ὁποῖον ἀποτελεῖ τήν κυρίαν τιμήν τοῦ ὁλοκληρώματος CAUCHY διά τά ἐσωτερικά σημεῖα t τῆς καμπύλης L εἶναι ἡ πλήρωσις ὑπό τῆς πυκνότητος $\varphi(t)$ τοῦ ὁλοκληρώματος CAUCHY (1) τῆς γενικευμένης συνθήκης HÖLDER (H^*) ἐπί τῆς καμπύλης L {MUSKHELISHVILI, 1953B, §29,§77}. Ταύτην θά θεωρῶμεν πληρουμένην πάντοτε ὑπό τῶν πυκνοτήτων $\varphi(t)$ ὁλοκληρωμάτων CAUCHY τῆς μορφῆς (1).

Περαιτέρω δύναται νά δειχθῇ ὅτι αἱ ὁριακαί τιμαί $\Phi^{\pm}(t)$ τῆς συναρτήσεως $\Phi(z)$ διά z τεῖνον εἰς ἕν σημεῖον t τῆς καμπύλης L ἐκ τῆς μιᾶς ἤ τῆς ἑτέρας πλευρᾶς (+ ἤ −) ταύτης ὑφίστανται ὑπό τάς προαναφερθείσας προϋποθέσεις καί πληροῦν τούς τύπους τοῦ PLEMELJ {MUSKHELISHVILI, 1953B, §17}:

$$\Phi^{+}(t) - \Phi^{-}(t) = \varphi(t) \quad , \quad \Phi^{+}(t) + \Phi^{-}(t) = 2\Phi(t) \quad , \tag{4}$$

ἔνθα ἡ συνάρτησις $\Phi(t)$ ὁρίζεται κατά τόν τύπον (3).

Διά τά προβλήματα τά ἐξεταζόμενα εἰς τήν παροῦσαν διατριβήν οὐδόλως ἐνδιαφέρουν αἱ τιμαί $\Phi(t)$ τῆς συναρτήσεως $\Phi(z)$ ἐπί τῆς καμπύλης L, ἀλλά μόνον αἱ ὁριακαί τιμαί $\Phi^{\pm}(t)$ τῆς συναρτήσεως $\Phi(z)$ παρά τάς πλευράς τῆς καμπύλης L. Ὡς ἐκ τούτου, θά ἠδύνατο νά θεωρηθῇ ὁ δεύτερος τύπος τοῦ PLEMELJ ὡς ὁρισμός τῆς συναρτήσεως $\Phi(t)$, ἥτις κατά ταῦτα ἰσοῦται μέ τόν μέσον ὅρον τῶν ὁριακῶν τιμῶν $\Phi^{+}(t)$ καί $\Phi^{-}(t)$, ὁπότε ὁ τύπος (3) θά ἠδύνατο νά ἀποδειχθῇ ὡς συνέπεια τοῦ δευτέρου τῶν τύπων (4).

Περαιτέρω χρησιμοποιοῦνται ἐνίοτε καί αἱ πρῶται παράγωγοι τῶν συναρτήσεων $\Phi(z)$ τῆς μορφῆς (1) διδόμεναι ὑπό τοῦ τύπου:

$$\Phi'(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{(\tau - z)^2}\,d\tau \quad , \tag{5}$$

ἐφ' ὅσον τό σημεῖον z δέν ἀνήκει εἰς τήν καμπύλην L. Εἰς τήν περίπτωσιν ταύτην ἱκανή συνθήκη ὑπάρξεως τῶν ὁριακῶν τιμῶν $\Phi'^{\pm}(t)$ παρά τήν καμπύλην L εἶναι ἡ πλήρωσις ὑπό τῆς παραγώγου $\varphi'(t)$ τῆς συναρτήσεως $\varphi(t)$ ὡς πρός t τῆς γενικευμένης συνθήκης HÖLDER (H^*) ἐπί τῆς καμπύλης L {GAKHOV, 1966, §4.4}. Ὅσον ἀφορᾷ εἰς τάς τιμάς $\Phi'(t)$ τῆς συναρτήσεως $\Phi'(z)$ διά τά σημεῖα t τῆς καμπύλης L, πλήν τῶν ἄκρων ταύτης, αὗται δύνανται νά ὁρισθοῦν βάσει τοῦ δευτέρου τῶν τύπων τοῦ PLEMELJ (4) μᾶλλον ἤ βάσει τύπου ἀναλόγου τοῦ (3), καθ' ὅσον ὁ τελευταῖος οὗτος δέν θά εἶχεν ἔννοιαν, πλήν ἐάν $\varphi(t) = 0$.