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On a method of numerical solution of a plane elasticity problem

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Abstract A numerical method for the solution of Cauchy type singular integral equations along contours in the complex plane is proposed. This method is based on a numerical integration rule for the evaluation of integrals of periodic functions along the real axis. Moreover, this method is applied to the Cauchy type singular integral equation to which the plane elasticity problem for a finite medium or an infinite medium with a hole can be reduced, after an appropriate modification of this equation assuring both the uniqueness of its solution and the automatic satisfaction of the condition of single-valuedness of displacements. An application of the proposed method to a medium having the form of an ellipse, or an infinite medium with an elliptical hole, is also made.

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1. Statement of the problem

The first fundamental (traction) problem of plane elasticity for a finite or infinite isotropic medium has a closed-form solution only in a few special cases (like the case when the medium has the shape of a circle). In general, this problem can be reduced to a Cauchy type complex singular integral equation or a Fredholm type complex integral equation along the boundary of the medium by a variety of techniques, some of which can be found in the book of Muskhelishvili [1].

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Fig. 1: Geometry of the finite medium D and the infinite medium \tilde{D} when their boundary L is an ellipse

One such technique, recently developed by Lardner [2], consists in considering a distribution of infinitesimal dislocations acting along the boundary L of the medium D (Fig. 1) and causing the tractions really applied on this boundary. If s or σ denotes a parameter changing along the boundary L of the medium with a period equal to 2π , $\tau(s)$ and $t(\sigma)$ denote the points of this boundary in complex form and $\theta(\sigma)$ denotes the angle between the tangent at a point $t(\sigma)$ of the boundary L and the Ox-axis, from the developments of Lardner [2] it can be shown that the problem reduces to the following Cauchy type singular integral equation:

$$\operatorname{Re} \int_{L} \frac{g_{2}(s) - ig_{1}(s)}{\tau(s) - t(\sigma)} \,\mathrm{d}s + ie^{2i\theta(\sigma)} \int_{L} \frac{g_{2}(s)\operatorname{Im}[\tau(s) - t(\sigma)] + g_{1}(s)\operatorname{Re}[\tau(s) - t(\sigma)]}{[\tau(s) - t(\sigma)]^{2}} \,\mathrm{d}s$$
$$= \pi[\sigma_{n}(\sigma) - i\sigma_{t}(\sigma)] \tag{1}$$

written here, for convenience, under a slightly modified form. In this equation, $\sigma_n(\sigma)$ and $\sigma_t(\sigma)$ denote the normal and shear components of tractions along the boundary *L* of the medium and $g(s) = g_1(s) + ig_2(s)$ denotes an unknown function proportional to the dislocation density along the boundary *L*.

Moreover, Eq. (1) should be supplemented by the conditions [2]

$$\int_{L} g_k(s) \,\mathrm{d}s = 0, \quad k = 1, 2, \tag{2}$$

which are necessary in the case of infinite media in order that the condition of single-valuedness of displacements holds. In the case of finite media, Eqs. (2) are necessary only for reasons of uniqueness of the solution of Eq. (1).

Once the complex singular integral equation (1) has been solved, both the problem of the finite simply-connected region D, surrounded by the boundary L, and the problem of the infinite region \tilde{D} , determined by the same boundary, will have also been solved, under the assumption that the tractions $\sigma_n(\sigma)$ and $\sigma_t(\sigma)$ have the same distribution along L for both these media D and \tilde{D} . Then the complex potentials $\Phi(z)$ and $\Psi(z)$ of plane elasticity [1, 3] are found to be given by

$$\Phi(z) = \frac{1}{2\pi i} \int_{L} \frac{g(s)}{\tau(s) - z} ds,$$

$$\Psi(z) = -\frac{1}{2\pi i} \int_{L} \frac{\overline{g(s)}}{\tau(s) - z} ds - \frac{1}{2\pi i} \int_{L} \frac{\overline{\tau(s)} g(s)}{[\tau(s) - z]^2} ds$$
(3)

either in the case of generalized plane stress or in the case of plane strain.

In this technical report, a method of numerical solution of Eq. (1) will be presented, based on the use of a modified form of the trapezoidal quadrature rule for the numerical evaluation of integrals of periodic functions.

2. On the numerical evaluation of contour integrals

For the numerical evaluation of a contour integral of a regular function, this function can be considered to be a periodic function of the real variable *s* characterizing the points of the contour. Then numerical integration rules for periodic functions can be successfully used [4]. Since the trapezoidal rule, properly modified, is very accurate when applied to the evaluation of integrals of periodic functions [4], we obtain after ignoring the error term

$$\int_{0}^{2\pi} f(s) \,\mathrm{d}s \approx \frac{\pi}{n} \sum_{i=0}^{2n-1} f(s_i), \quad s_i = \frac{i\pi}{n},$$
(4)

where the variable *s* is assumed, without loss of generality, to vary in the interval $[0, 2\pi]$ along the whole contour *L*.

In the case when we have to evaluate Cauchy type principal value integrals along the contour L, we can use the following formula derived by Chawla and Ramakrishnan [5] for Cauchy type principal value integrals of periodic functions and also based on the trapezoidal quadrature rule:

$$\int_0^{2\pi} f(s) \cot \frac{s-\sigma}{2} ds \approx \frac{\pi}{n} \sum_{i=0}^{2n-1} f(s_i) \cot \frac{s_i-\sigma}{2} + 2\pi f(\sigma) \cot n\sigma, \quad s_i = \frac{i\pi}{n}.$$
 (5)

Now there results that if $\cot n\sigma = 0$, that is

$$\sigma = \sigma_k = \frac{(2k-1)\pi}{2n},\tag{6}$$

then Eq. (4) is valid not only in the case of regular integrals, but also in the case of Cauchy type principal value integrals.

In this way, we can select the points σ_k of application of Eq. (1) so as to be able to apply Eq. (4) to the integrals of this equation and to reduce it to a system of linear equations, which, when solved, gives the values of the unknown functions $g_1(s)$ and $g_2(s)$ at the points $s_i = i\pi/n$ used in Eqs. (4) and (5).

3. On the uniqueness of solution of the integral equation (1)

The complex singular integral equation (1) has not a unique solution. This is due to two reasons:

(i) The complex potential $\Phi(z)$, given by the first of Eqs. (3), can be changed by a constant imaginary value with no change in the stress field in the media D or \tilde{D} [1]. This means that $g(s)/\tau'(s)$ can also change by a constant imaginary value, the boundary conditions on L, expressed by Eq. (1), remaining valid. Following Theocaris and Tsamasphyros [6], we can add to the left-hand side of this equation a term of the form $C_1 \exp[2i\theta(\sigma)] b_1/t^2(\sigma)$, where C_1 is an arbitrary not purely real constant and the quantity b_1 is given by

$$b_1 = \operatorname{Im} \int_L \frac{g_2(s) - ig_1(s)}{\tau(s)} \,\mathrm{d}s.$$
(7)

Then this reason of non-uniqueness of the solution of Eq. (1) is no more valid. When writing Eq. (7), we also assume that the origin of coordinates *O* lies inside the medium *D*.

(ii) Conditions (2) of single-valuedness of displacements are not incorporated into Eq. (1). This causes this equation not to have a unique solution. Of course, these conditions can be taken into account as independent conditions supplementing Eq. (1) when trying to solve it numerically. But this technique has the disadvantage, when used, that one of the points σ_k of application of Eq. (1), given by Eq. (6), should be ignored in order that conditions (2) be taken into account. For this reason, it is obvious that conditions (2) should be incorporated into Eq. (1) and, to achieve this, it is enough to add one more term of the form $C_2 \exp[2i\theta(\sigma)]b_2/t(\sigma)$ to the left-hand side of Eq. (1), where C_2 is an arbitrary constant and the quantity b_2 is given by

$$b_2 = \int_L [g_2(s) - ig_1(s)] \,\mathrm{d}s. \tag{8}$$

We also take into consideration that Eq. (1), because of Eqs. (3), is simply an expression of the boundary conditions along L

$$\Phi^{\pm}(t) + \overline{\Phi^{\pm}(t)} + \frac{\mathrm{d}t}{\mathrm{d}t} \left[\bar{t} \Phi'^{\pm}(t) + \Psi^{\pm}(t) \right] = \overline{p(t)}, \quad t \equiv t(\sigma), \tag{9}$$

where

$$p(t) = \sigma_n(t) + i\sigma_t(t), \qquad (10)$$

satisfied by the boundary values of the complex potentials $\Phi(z)$, $\Phi'(z)$ and $\Psi(z)$ when *z* tends to a point $t(\sigma)$ of the boundary *L* from *D* or \tilde{D} . It is also assumed that both media *D* and \tilde{D} loaded by the tractions $\sigma_n(\sigma)$ and $\sigma_t(\sigma)$ are in equilibrium. This means that the total force and the total moment caused by $\sigma_n(\sigma)$ and $\sigma_t(\sigma)$ along the whole boundary *L* are equal to zero. Then we have

$$\int_{L} \overline{p(t)} \,\overline{\mathrm{d}t} = 0 \tag{11}$$

and

$$\operatorname{Re} \int_{L} \overline{p(t)} t \, \overline{\mathrm{d}t} = 0 \tag{12}$$

respectively.

When we add the two terms mentioned previously to Eq. (1), this equation takes the form

$$\operatorname{Re} \int_{L} \frac{g_{2}(s) - ig_{1}(s)}{\tau(s) - t(\sigma)} \,\mathrm{d}s + ie^{2i\theta(\sigma)} \int_{L} \frac{g_{2}(s)\operatorname{Im}[\tau(s) - t(\sigma)] + g_{1}(s)\operatorname{Re}[\tau(s) - t(\sigma)]}{[\tau(s) - t(\sigma)]^{2}} \,\mathrm{d}s + e^{2i\theta(\sigma)} \Big[\frac{C_{1}b_{1}}{t^{2}(\sigma)} + \frac{C_{2}b_{2}}{t(\sigma)} \Big] = \pi[\sigma_{n}(\sigma) - i\sigma_{t}(\sigma)],$$
(13)

which, because of Eqs. (3) and (10) and the Plemelj formulae, gives

$$\Phi^{+}(t) + \overline{\Phi^{+}(t)} + \frac{\mathrm{d}t}{\mathrm{d}t} \left[\overline{t} \Phi^{\prime +}(t) + \Psi^{+}(t) \right] + \frac{1}{\pi} \frac{\mathrm{d}t}{\mathrm{d}t} \left[\frac{C_1 b_1}{t^2} + \frac{C_2 b_2}{t} \right] = \overline{p(t)}, \quad t \equiv t(\sigma).$$
(14)

Now, by substituting $\overline{p(t)}$ in Eqs. (11) and (12) and taking into account the fact that for a complex function f(z) analytic in a simply-connected domain D surrounded by a closed contour L it is valid that

$$\int_{L} f(t) \,\mathrm{d}t = 0,\tag{15}$$

we can show that the constants b_2 and b_1 respectively should vanish.

In this way, when solving Eq. (13) numerically, we have neither to take into account conditions (2) nor to have doubts on the uniqueness of its solution if the remarks made in Ref. [6] are also taken into account.

4. On the numerical solution of the integral equation (13)

As regards the numerical solution of the complex singular integral equation (13), at first we can write it under the form of two real singular integral equations. Afterwards, if we apply it at the points σ_k (k = 0, 1, ..., 2n - 1) of the contour L (given by Eq. (6)) and approximate the integrals involved in it by using Eq. (4), we obtain a system of 4n linear equations. When using Eq. (4), we take into account that it is valid both for regular and for Cauchy type principal value integrals for the selected points σ_k . After having solved this system of linear equations and determined the values of the unknown functions $g_1(s)$ and $g_2(s)$ at the points $s_i = i\pi/n$ (i = 0, 1, ..., 2n - 1), we can find approximate expressions for these functions along the whole boundary L, by using interpolation methods, or approximate the complex potentials $\Phi(z)$ and $\Psi(z)$, by using Eq. (4) once more. Thus, the whole problem will have been solved and the components of stresses and displacements at any point of the media D and \tilde{D} , their boundary L included, could be easily obtained.

On this point we can note that, even if the medium we are interested in is the finite medium D (or the infinite medium \tilde{D}), when solving Eq. (13), we obtain at the same time the solution of the plane elasticity problem for both media D and \tilde{D} assumed loaded along their boundary L in the same way.

It is also evident that the method of numerical solution of Cauchy type singular integral equations along contours, presented previously, can be applied not only to the first fundamental problem of plane elasticity for a simple medium but also to more complicated problems of plane elasticity which can also be reduced to Cauchy type singular integral equations, as well as to a lot of other analogous engineering problems.

Finally, it may be noted that the method of numerical solution of Cauchy type singular integral equations along contours used in this technical report is, in most cases, more accurate than the classical method of solution of such equations, which consists in considering the integrands as constants in each one of the intervals in which the integration interval (that is the contour) is divided by two consecutive abscissae and neglecting the contribution from the interval in which the Cauchy type singularity occurs (see, e.g., Ref. [7]).

5. Application to the case of ellipses

As an application, we will consider the problem of an ellipse D (or an infinite medium with an elliptical hole \tilde{D}) as shown in Fig. 1. The boundary L of the ellipse is assumed to be loaded by a constant normal pressure p. In the case of an infinite medium with an elliptical hole, such a loading is equivalent to a uniform loading at infinity. This application has been selected because it is one of a few cases when a closed-form solution can be easily found.

In this way, it is easy to see from Eq. (14) and after taking into account Eq. (10) that the complex potentials $\Phi(z)$ and $\Psi(z)$ can be expressed for the ellipse *D* as

$$\Phi(z) = -\frac{p}{2}, \quad \Psi(z) = 0, \quad z \in D.$$
(16)

Moreover, if a and b denote the semiaxes of the ellipse $(a \ge b)$, its points can be represented as

$$\tau(s) = R(e^{is} + me^{-is}) \quad \text{or} \quad t(\sigma) = R(e^{i\sigma} + me^{-i\sigma}), \quad 0 \le s, \sigma < 2\pi,$$
(17)

where

$$R = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b}.$$
(18)

In this case, from the solution of the problem of an infinite medium with an elliptical hole \tilde{D} under constant normal pressure *p* along the boundary *L* of the elliptical hole given in Ref. [3] it

Values of $g_1(s)/(ap\sin s)$					
b/a	2n = 4	2n = 8	2n = 16	2n = 32	Theoretical value
0.1	0.89217462	0.38218670	0.16012648	0.081083061	0.05
0.3	0.19751234	0.15074077	0.15000787	0.15000007	0.15
0.5	0.25181500	0.25000056	0.25000000	0.25000000	0.25
0.7	0.35002131	0.35000000	0.35000000	0.35000000	0.35
0.9	0.45000000	0.45000000	0.45000000	0.45000000	0.45
Values of $-g_2(s)/(ap\cos s)$					
b/a	2n = 4	2n = 8	2n = 16	2n = 32	Theoretical value
0.1	0.60301766	0.50670922	0.49326825	0.49722033	0.5
0.3	0.48762904	0.49977819	0.49999764	0.49999998	0.5
0.5	0.49909619	0.49999972	0.50000000	0.50000000	0.5
0.7	0.49998508	0.50000000	0.50000000	0.50000000	0.5
0.9	0.50000000	0.50000000	0.50000000	0.50000000	0.5

Table 1: Comparison of numerical and theoretical results

 for the problem of ellipses under a constant pressure

is easy to see that the value $\Phi^{-}(t)$ of the complex potential $\Phi(z)$ at the points $t = t(\sigma)$ of the boundary *L* will be given by

$$\Phi^{-}(t) = \frac{pme^{-i\sigma}}{e^{i\sigma} - me^{-i\sigma}}.$$
(19)

By taking into account the Plemelj formulae as well as the first of Eqs. (3) and Eqs. (17), (18) and (19), we can easily find that

$$g_1(s) = \frac{bp}{2}\sin s, \quad g_2(s) = -\frac{ap}{2}\cos s$$
 (20)

or, in a slightly different form,

$$\frac{g_1(s)}{ap} = \frac{1}{2} \frac{b}{a} \sin s, \quad \frac{g_2(s)}{ap} = -\frac{1}{2} \cos s.$$
(21)

In Table 1 we give the values of $g_1(s)/(ap\sin s)$ and $-g_2(s)/(ap\cos s)$ found after the numerical solution of Eq. (13) with $C_1 = i$, $C_2 = 1$ and for different values of the ratio b/a and of the number of points 2n used, compared with their theoretical values. We observe that the obtained numerical results, with no symmetry in the geometry and loading having been taken into consideration, are good enough especially when the ratio b/a tends to unity, that is the ellipse tends to become a circle. In this special case, no approximation is involved in numerical integrations.

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 $^{^{3}}$ All the links (external links in blue) in this section were added by the first author on 25 February 2018 for the online publication of this technical report.