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A new method for the solution of the problem of a star-shaped crack in an infinite isotropic elastic medium

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Abstract The problem of a symmetrical and symmetrically loaded star-shaped crack in an infinite isotropic elastic medium and under conditions of generalized plane stress or plane strain can be solved by reduction to a complex Cauchy type singular integral equation on one of the branches of the crack. The numerical solution of this equation is achieved by its reduction to a system of linear algebraic equations resulting by its application at appropriately selected points of the integration interval and the subsequent use of a modified form of the Gauss numerical integration rule for the approximation of the integrals appearing in this equation.

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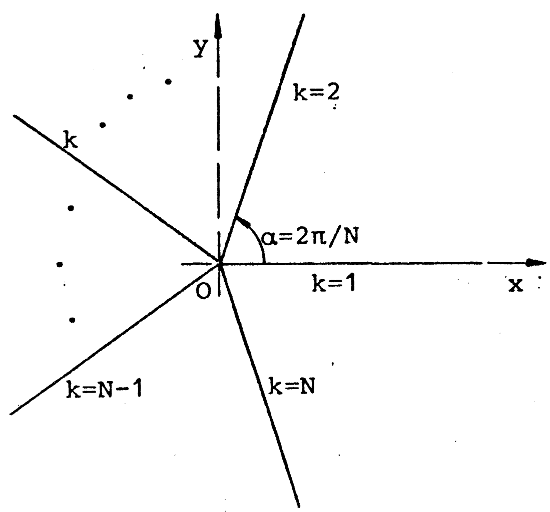

Fig. 1: A star-shaped crack in an infinite medium

1. Introduction

The problem of a symmetrical star-shaped crack with N branches of length a (as is the crack of Fig. 1) lying in an infinite isotropic elastic medium under generalized plane stress or plane strain conditions and loaded in the same way on all its branches has been already investigated by using different methods.

Thus, Westmann [1] solved the above problem in the case of a constant pressure distribution on the branches of the crack. Next, Srivastav and Narain [2] solved the same problem in the case of an arbitrary pressure distribution on the branches of the crack by using the Mellin transform. Westmann [1] also used the Wiener–Hopf technique for the determination of the stress intensity factor k_{I} at the end of each branch of the crack whereas Srivastav and Narain [2] reduced the whole problem to the solution of a Fredholm integral equation of the second kind. These methods of solution of the problem of a star-shaped crack are also described in Refs. [3] and [4]. The case of an arbitrary pressure distribution on the branches of the star-shaped crack was also studied by Williams [5] by using the Wiener–Hopf technique. Finally, for the solution of the problem under consideration Kutter [6] proposed the use of the method of complex potentials of Muskhelishvili in combination with the conformal mapping of the star-shaped crack of Fig. 1 on the unit circle.

Although all the aforementioned methods of solution of the plane problem of the star-shaped crack in an infinite isotropic elastic medium lead to the correct solution of this problem and to the determination of the stress intensity factor k_{I} at the tips of the branches of the crack with a satisfactory accuracy, nevertheless, they present the disadvantages on the one hand that they make use of complicated algebraic transformations and on the other hand that by no means are they generalizable to the case where the branches of the star-shaped crack are not straight but curvilinear.

Here a direct, easy-to-use and very efficient method of solution of the problem under consideration is proposed. This method is based on the method of reduction of the plane problem of a system of straight cracks in an infinite isotropic elastic medium to a system of Cauchy type singular integral equations proposed by Datsyshin and Savruk [7, 8]. Although Datsyshin and Savruk considered only the case where the straight cracks do not have common points [7, 8], nevertheless, it is easily observed that even in the case where the straight cracks under consideration have a common point (as is the case in Fig. 1 with the branches of the star-shaped crack shown there), the method proposed by them remains valid. Finally, the whole problem is reduced to a complex Cauchy type singular integral equation on one of the branches of the crack, which can be numerically solved by using a method based on the Gauss numerical integration rule for the first time proposed by the

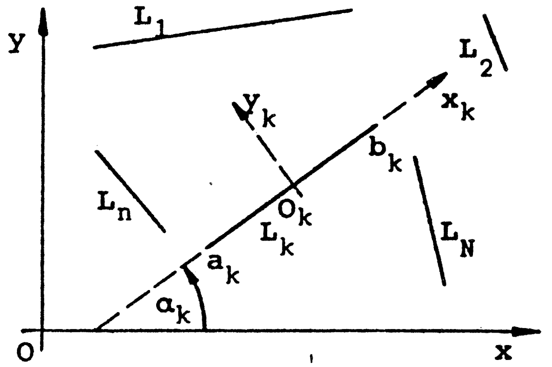

Fig. 2: A system of straight cracks in an infinite medium

authors [9].

Finally, it can be remarked that in the case where the branches of the star-shaped crack of Fig. 1 are curvilinear, the method presented here can easily be generalized to this case by taking into account the approach described in Ref. [10] for the construction of the singular integral equation to be solved but with the numerical technique used here remaining unchanged.

2. The singular integral equation

For the reduction of the problem of the star-shaped crack of Fig. 1 to a singular integral equation the star-shaped crack can be considered consisting of N straight and concurrent cracks, which are its branches.

The use of the method of reduction of the plane problem of a system of N straight cracks L_n in an infinite plane isotropic elastic medium (Fig. 2) to a system of singular integral equations proposed by Datsyshin and Savruk [7, 8] is already possible. Hence, the system of cracks of Fig. 2, each of which is assumed loaded by a normal and a shear loading in an arbitrary but the same way on both its edges, leads to the following system of Cauchy type singular integral equations [7, 8]:

$$\sum_{k=1}^{N} \int_{a_k}^{b_k} \left[g_k'(t) K_{nk}(t,x_n) + \overline{g_k'(t)} L_{nk}(t,x_n) \right] \mathrm{d}t = \pi[\sigma_n(x_n) - i\tau_n(x_n)],$$
$$a_n < x_n < b_n, \quad n = 1,2,\ldots,N. \tag{1}$$

Here $g_k'(t)$ ($k = 1,2,\ldots,N$) are the unknown functions to be determined on the N cracks. These functions can be assumed proportional to the dislocation densities on the cracks L_n. Next, $\sigma_n(x_n)$ and $\tau_n(x_n)$ are known functions denoting the distributions of the normal and shear loadings, respectively, on the nth ($n = 1,2,\ldots,N$) crack in a local coordinate system $O_n x_n y_n$ (Fig. 2) attached to the crack L_n. Finally, the kernels $K_{nk}(t,x)$ and $L_{nk}(t,x)$ are known and they are given by the relations [7, 8]

$$K_{nk}(t,x) = \frac{e^{i\alpha_k}}{2} \left[\frac{1}{T_k - X_n} + \frac{e^{-2i\alpha_n}}{\bar{T}_k - \bar{X}_n} \right], \tag{2}$$

$$L_{nk}(t,x) = \frac{e^{-i\alpha_k}}{2} \left[\frac{1}{\bar{T}_k - \bar{X}_n} - \frac{T_k - X_n}{(\bar{T}_k - \bar{X}_n)^2} e^{-2i\alpha_n} \right], \tag{3}$$

where α_k ($k = 1,2,\ldots,N$) is the angle formed between the Ox-axis and the Ox_k-axis (the direction of the kth crack), as is shown in Fig. 2, and T_k and X_n are the quantities

$$T_k := t e^{i\alpha_k} + z_k^0, \quad X_n := x e^{i\alpha_n} + z_n^0, \quad k,n = 1,2,\ldots,N. \tag{4}$$

In these last relations, we have put

$$z_k^0 := x_k^0 + iy_k^0, \quad k = 1, 2, \ldots, N, \tag{5}$$

where x_k^0 and y_k^0 are the coordinates of the point O_k (the centre of the local coordinate system $O_k x_k y_k$) with respect to the global coordinate system Oxy (Fig. 2).

Moreover, in the case of N separate straight cracks (as is the case in Fig. 2), the system of singular integral equations (1) should be supplemented by N integral conditions assuring the uniqueness of single-valuedness of displacements around each crack. Nevertheless, it is evident that for the star-shaped crack shown in Fig. 1 because of the assumed symmetry both in the geometry and in the loading, the condition of single-valuedness of displacements around the whole star-shaped crack is automatically satisfied and, therefore, we do not have to take it into account here.

As far as the complex potentials $\Phi(z)$ and $\Psi(z)$ of Muskhelishvili [11] are concerned, on the basis of which every stress or strain quantity can be determined on the whole cracked plane [11], after the determination of the unknown functions $g_k'(t)$ $(k = 1, 2, \ldots, N)$ these potentials are expressed as [8]

$$\Phi(z) = \frac{1}{2\pi} \sum_{k=1}^{N} \int_{a_k}^{b_k} \frac{g_k'(t)}{t - z_k} \, \mathrm{d}t, \quad z_k = x_k + iy_k, \tag{6}$$

$$\Psi(z) = \frac{1}{2\pi} \sum_{k=1}^{N} e^{-2i\alpha_k} \int_{a_k}^{b_k} \Big[\frac{\overline{g_k'(t)}}{t - z_k} - \frac{\bar{T}_k e^{i\alpha_k} g_k'(t)}{(t - z_k)^2} \Big] \mathrm{d}t, \quad z_k = x_k + iy_k. \tag{7}$$

At this point we should mention that another method to tackle the problem of interacting straight cracks in an infinite plane isotropic elastic medium, which is similar to the aforementioned method of Datsyshin and Savruk [7, 8], was proposed by Pučik [12]

Now we can apply the above approach to the star-shaped crack of Fig. 1 assuming it consisting of its N straight branches. Because of the symmetry in geometry and in loading, it is obvious that the functions $g_k'(t)$ will have the same distribution on all the branches of the crack. Therefore, they can be considered as a single unknown function $g'(t)$. Moreover, the satisfaction of one of the singular integral equations (1), let us assume that concerning the branch with $k = 1$ lying on the Ox-axis (Fig. 1), ensures the satisfaction of the remaining $N - 1$ singular integral equations (1) on the remaining branches of the crack.

In the case of the star-shaped crack under consideration, it is also appropriate to choose the local centres O_k of the coordinate systems on each branch of the star-shaped crack so that they coincide with the apex O of the star-shaped crack, which is also the centre of the global coordinate system Oxy. Then we have

$$\alpha_k = \frac{2\pi}{N}(k-1), \quad T_k = t\varepsilon^{k-1}, \quad X_n = x\varepsilon^{n-1} \quad \text{with} \quad \varepsilon := e^{2\pi i/N}. \tag{8}$$

Moreover, the kernels $K_{nk}(t,x)$ and $L_{nk}(t,x)$ appearing in the system of singular integral equations (1) and determined on the basis of Eqs. (2) and (3), respectively, will be given by the formulae

$$K_{nk}(t,x) = \frac{1}{2} \Big[\frac{1}{t - x\varepsilon^{n-k}} + \frac{\varepsilon^{2(k-n)}}{t - x\varepsilon^{k-n}} \Big], \tag{9}$$

$$L_{nk}(t,x) = \frac{1}{2} \Big[\frac{1}{t - x\varepsilon^{k-n}} - \frac{t - x\varepsilon^{n-k}}{(t - x\varepsilon^{k-n})^2} \varepsilon^{2(k-n)} \Big]. \tag{10}$$

Now, as was described above, we have to solve the singular integral equation

$$\int_0^a [g'(t) K(t,x) + \overline{g'(t)} L(t,x)] \, \mathrm{d}t = \pi[\sigma(x) - i\tau(x)], \quad 0 < x < a. \tag{11}$$

Here a is the length of each branch of the star-shaped crack, $\sigma(x)$ and $\tau(x)$ are its normal and shear loading, respectively, and the kernels $K(t,x)$ and $L(t,x)$ can result from the formulae

$$K(t,x) = \sum_{k=1}^{N} K_{nk}(t,x), \quad L(t,x) = \sum_{k=1}^{N} L_{nk}(t,x). \tag{12}$$

We further take into account the formulae [13]

$$\sum_{k=0}^{N-1} \frac{1}{t - x\varepsilon^k} = \frac{Nt^{N-1}}{t^N - x^N}, \quad \sum_{k=0}^{N-1} \frac{\varepsilon^{2k}}{t - x\varepsilon^k} = \frac{Ntx^{N-2}}{t^N - x^N} \tag{13}$$

as well as the formulae resulting by a differentiation of the first of these formulae with respect to x and of the second of these formulae with respect to t. Then, on the basis of Eqs. (9), (10) and (12), we find that the singular integral equation (11) takes the form

$$\int_0^a \left\{ g'(t)\, \frac{t(t^{N-2} + x^{N-2})}{t^N - x^N} + \overline{g'(t)} \left[\frac{t(t^{N-2} + x^{N-2})}{t^N - x^N} - \frac{Nt^{N-1}x^{N-2}(t^2 - x^2)}{(t^N - x^N)^2} \right] \right\} \mathrm{d}t = \frac{2\pi}{N}\,[\sigma(x) - i\tau(x)]. \tag{14}$$

Next, if the unknown function $g'(t)$ is separated into its real and its imaginary parts, i.e.

$$g'(t) = g_1'(t) + ig_2'(t), \tag{15}$$

the complex singular integral equation (14) is reduced to the following system of two real singular integral equations:

$$\int_0^a g_1'(t)\, \frac{t}{t^N - x^N} \left[t^{N-2} + x^{N-2} - \frac{Nt^{N-2}x^{N-2}(t^2 - x^2)}{2(t^N - x^N)} \right] \mathrm{d}t = \frac{\pi}{N}\,\sigma(x), \quad 0 < x < a, \tag{16}$$

$$\int_0^a g_2'(t)\, \frac{Nt^{N-1}x^{N-2}(t^2 - x^2)}{2(t^N - x^N)^2}\, \mathrm{d}t = -\frac{\pi}{N}\,\tau(x), \quad 0 < x < a. \tag{17}$$

From these equations we observe that when there exists only a normal loading $\sigma(x)$ along the branches of the star-shaped crack, the sought function $g'(t)$ is purely real. On the contrary, when there exists only a shear loading $\tau(x)$, this function $g'(t)$ is purely imaginary.

After the determination of the function $g'(t) = g_1'(t) + ig_2'(t)$ from the solution of the system of singular integral equations (16) and (17), the complex potentials $\Phi(z)$ and $\Psi(z)$ can be determined from Eqs. (6) and (7), respectively. For the problem of the star-shaped crack under consideration, taking also into account Eqs. (13), we find that these formulae take the simpler forms

$$\Phi(z) = \frac{N}{2\pi} \int_0^a g'(t)\, \frac{t^{N-1}}{t^N - z^N}\, \mathrm{d}t, \tag{18}$$

$$\Psi(z) = \frac{N}{2\pi} \int_0^a \left[\overline{g'(t)} - g'(t)\, \frac{(N-1)t^N + z^N}{t^N - z^N} \right] \frac{tz^{N-2}}{t^N - z^N}\, \mathrm{d}t. \tag{19}$$

Finally, we can mention that for the determination of the complex stress intensity factor k^* at the tips of the branches of the star-shaped crack we can use the formula [7]

$$k^* = k_{\mathrm{I}} - ik_{\mathrm{II}} = -\sqrt{2} \lim_{t \to a} [\sqrt{a - t}\, g'(t)]. \tag{20}$$

Because of Eq. (15), this formula is separated into the formulae

$$k_{\mathrm{I}} = -\sqrt{2} \lim_{t \to a} [\sqrt{a - t}\, g_1'(t)], \quad k_{\mathrm{II}} = \sqrt{2} \lim_{t \to a} [\sqrt{a - t}\, g_2'(t)]. \tag{21}$$

Table 1

Numerical results for the reduced stress intensity factor $k_{\mathrm{I}}/(\sigma\sqrt{a})$ in the case of loading of the star-shaped crack of Fig. 1 (with N branches) by a constant normal compressive loading σ by using the singular integral equation (16) and $m=5$ linear algebraic equations

N	$\dfrac{k_{\mathrm{I}}}{\sigma\sqrt{a}}$	Approximate formula [1] for $N\to\infty$: $k_{\mathrm{I}}/(\sigma\sqrt{a})\approx 2/\sqrt{N}$	Values of $k_{\mathrm{I}}/(\sigma\sqrt{a})$ determined by other methods
2	0.99989		1.0000 (theoretical value)
3	0.94178		0.9419 [5]
4	0.86366		0.8641 [5]
5	0.79733		0.7976 [5]
6	0.74286		0.7425 [5]
8	0.65889		0.6590 [5]
10	0.59741	0.63246	≈ 0.61 [1]
15	0.49906	0.51640	
20	0.43965	0.44721	
30	0.36179	0.36515	

3. The numerical solution

For the numerical solution of the real singular integral equations (16) and (17) one can use the method proposed by the authors and described in Ref. [9]. With the help of this method by using a modified form of the Gauss numerical integration rule and an appropriate selection of the collocation points in the singular integral equation to be solved, this equation can be reduced to a system of linear algebraic equations.

For the application of the aforementioned method the fact that the unknown functions $g'_{1,2}(t)$ present singularities of order $-1/2$ near the tips of the branches of the star-shaped crack [7, 8, 14] is also taken into account. Near the apex O of this crack the unknown function $g'_1(t)$ does not present a singularity as can easily be observed [14]. Therefore, because of the symmetry of the problem as well, it tends to zero at this apex.

On the contrary, for the unknown function $g'_2(t)$ the case when this function presents a logarithmic singularity near the apex O of the star-shaped crack cannot be excluded [14]. Such a singularity can be ignored if we are interested only in the value of the stress intensity factor k_{II} at the tips of the branches of the crack. In the opposite case, the logarithmic singularity should be taken into account. For this task it is required that instead of the ordinary Gauss rule used in Ref. [9], a modified form of the Gauss rule should be used in agreement with the developments of Ref. [15].

In Table 1, numerical results for the reduced stress intensity factor $k_{\mathrm{I}}/(\sigma\sqrt{a})$ are displayed in the case of loading of the star-shaped crack of Fig. 1 by a constant normal compressive loading σ. These results resulted by a reduction of the singular integral equation (16) to a system of $m=5$ linear algebraic equations and they are in a satisfactory agreement with their values already having been determined by other methods as is clear from Table 1.

Finally, it can be noted that instead of the Gauss method, the Lobatto method could also be used for the numerical integrations [9]. In this case, the condition $g'_1(0)=0$ must also be taken into account for the completion of the system of linear algebraic equations.

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[3] All the links (external links in blue) in this section were added by the second author on 24 April 2018 for the online publication of this technical report. Moreover, final publication details were added in Refs. [10, 15].

Summary

Several methods for the solution of the problem of a symmetrical and symmetrically loaded star-shaped crack in an infinite plane isotropic elastic medium under conditions of generalized plane stress or plane strain have been developed up to now [1–5]. These methods present the disadvantages of requiring lengthy calculations, due to the transform or conformal mapping techniques used, and of being inefficient to treat the case of curvilinear branches of the star-shaped crack.

Here a new method, based on the developments of Refs. [7, 8], is proposed which reduces, in a direct manner, the whole problem to a complex Cauchy type singular integral equation on one of the branches of the crack. This method can easily be generalized to the case of curvilinear crack branches if the results of Ref. [10] are taken into account. After the numerical solution of the complex Cauchy type singular integral equation, the complex potentials $\Phi(z)$ and $\Psi(z)$ of Muskhelishvili [11] can easily be determined.

As regards the numerical solution of Cauchy type singular integral equations, the method proposed in Ref. [9] can be successfully used. This method, developed by the authors, consists in the reduction of a Cauchy type singular integral equation to a system of linear algebraic equations by applying it at a number of appropriately selected points of the integration interval and using a modified form of the Gauss (or Lobatto) numerical integration rule for the approximation of the integrals. As an application, the case of a uniform pressure distribution on the crack branches is examined. The values of the stress intensity factors obtained in this case coincide with the results obtained by other methods.